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Liouvillian solutions of first order nonlinear differential equations

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A B S T R A C T

Let k be a differential field of characteristic zero and E be a liouvillian extension of k. For any differential subfield K intermediate to E and k, we prove that there is an element in the set $K - k$ satisfying a linear homogeneous differential equation over k. We apply our results to study liouvillian solutions of first order nonlinear differential equations and provide generalisations and new proofs for several results of M. Singer and M. Rosenlicht on this topic.

1. Introduction

Throughout this article, we fix a differential field k of characteristic zero. Let E be a differential field extension of k and let $'$ denote the derivation on E. We say that E is a *liouvillian extension*[1] of k if $E = k(t_1, \cdots, t_n)$ and there is a tower of differential fields

$$k = k_0 \subseteq k_1 \subseteq \cdots \subseteq k_n = E$$

such that for each i, $k_i = k_{i-1}(t_i)$ and either $t_i' \in k_{i-1}$ or $t_i'/t_i \in k_{i-1}$ or t_i is algebraic over k_{i-1}. If $E = k(t_1, \cdots, t_n)$ is a liouvillian extension such that $t_i' \in k_{i-1}$ for each i then we call E an *iterated antiderivative extension* of k. A solution of a differential equation over k is said to be liouvillian over k if the solution belongs to some liouvillian extension of k.

Let $\mathcal{P}$ be an $n + 1$ variable polynomial over k. We are concerned with the liouvillian solutions of the differential equation

$$\mathcal{P}(y, y', \cdots, y^{(n)}) = 0. \tag{1.1}$$

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[1] We adopt Rosenlicht's definition of a liouvillian extension (see [4], page 371), which in turn is equivalent to the one given by Kolchin in [2], page 408, exercise 5. For details, read Remark 1.4.

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We prove in Theorem 2.2 that

> if E is a liouvillian extension of k and K is a differential field intermediate to E and k then $K = k\langle u_1, \cdots, u_l\rangle$ where for each i, the element u_i satisfies a linear differential equation over $k\langle u_1, \cdots, u_{i-1}\rangle$. Moreover if E is an iterated antiderivative extension of k having the same field of constants as k then each u_i can be chosen so that $u_i' \in k(u_1, \cdots, u_{i-1})$, that is, K is also an iterated antiderivative extension of k.

Our result regarding iterated antiderivative extensions generalises the main result of [8] to differential fields k having a non-algebraically closed field of constants. The main ingredient used in the proof of our theorem is Lemma 2.1 using which we also obtain the following interesting results concerning solutions of nonlinear differential equations.

A. In Remark 2.3, we show that if E is an iterated antiderivative extension of k having the same field of constants and if $y \in E$ and $y \notin k$ satisfies a differential equation $y' = \mathcal{P}(y)$, where $\mathcal{P}$ is a polynomial in one variable over k, then degree of $\mathcal{P}$ must be less than or equal to 2.

B. Let C be an algebraically closed field of characteristic zero with the trivial derivation. In Proposition 3.1, we prove that for any rational function $\mathcal{R}$ in one variable over C, the differential equation $y' = \mathcal{R}(y)$ has a non-constant liouvillian solution y if and only if $1/\mathcal{R}(y)$ is of the form $\partial z/\partial y$ or $(1/az)(\partial z/\partial y)$ for some $z \in C(y)$ and for some nonzero element $a \in C$. This result generalises and provides a new proof for a result of Singer (see [5], Corollary 2). We also prove that for any polynomial $\mathcal{P}$ in one variable over C such that the degree of $\mathcal{P}$ is greater than or equal to 3 and that $\mathcal{P}$ has no repeated roots, the differential equation $(y')^2 = \mathcal{P}(y)$ has no non-constant liouvillian solution over C. This result appears as Proposition 3.2 and it generalises an observation made by Rosenlicht [4] concerning non-constant liouvillian solutions of the elliptic equation $(y')^2 = y^3 + ay + b$ over complex numbers with a nonzero discriminant.

C. Using Theorem 2.2 one can construct a family of differential equations with only algebraic solutions: Let $\alpha_2, \alpha_3, \cdots, \alpha_n \in k$ such that $x' \neq \alpha_2$ and $x' \neq \alpha_3$ for any $x \in k$. Let $\overline{k}$ be an algebraic closure of k and let E be a liouvillian extension of $\overline{k}$ with $C_E = C_{\overline{k}}$. We prove in Proposition 3.3 that if there is an element $y \in E$ such that

$$y' = \alpha_n y^n + \cdots + \alpha_3 y^3 + \alpha_2 y^2$$

then $y \in \overline{k}$.

In a future publication, the author hopes to develop the techniques in this paper further to provide an algorithm to solve the following problem, which appears as "Problem 7" in [7]: Give a procedure to decide if a polynomial first order differential equation $\mathcal{P}(y, y') = 0$, over the ordinary differential field $\mathbb{C}(x)$ with the usual derivation d/dx, has an elementary solution and to find one if it does.

Preliminaries and notations. A *derivation* of the field k, denoted by $'$, is an additive endomorphism of k that satisfies the Leibniz law $(xy)' = x'y + xy'$ for every $x, y \in k$. A field equipped with a derivation map is called a *differential field*. For any $y \in k$, we will denote first and second derivatives of y by y' and y'' respectively and for $n \geq 3$, the nth derivative of y will be denoted by $y^{(n)}$. The set of *constants* C_E of a differential field E is the kernel of the endomorphism $'$ and it can be seen that the set of constants is a differential subfield of k. Let E and k be differential fields. We say that E is a *differential field extension* of k if E is a field extension of k and the restriction of the derivation of E to k coincides with the derivation of k. Whenever we write $k \subseteq E$ as differential fields, we mean that E is a differential field extension of k and we write $y \in E - k$