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A note on strong protomodularity, actions and quotients

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ABSTRACT

In order to study the problems of extending an action along a quotient of the acted object and along a quotient of the acting object, we investigate some properties of the fibration of points. In fact, we obtain a characterization of protomodular categories among quasi-pointed regular ones, and, in the semi-abelian case, a characterization of strong protomodular categories. Eventually, we return to the initial questions by stating the results in terms of internal actions.

1. Introduction

The present work originates from the investigation of the categorical properties related to two well-known features of group actions.

Actions on quotients

Suppose we are given a pair (ξ, g):

$$A \times Y \overset{\xi}{\rightsquigarrow} Y \xrightarrow{g} Z\,,$$

where ξ is a left-action of groups, and g is a surjective homomorphism[1]. We discuss the following problem: under what conditions does the action ξ induces an action on the quotient Z?

Indeed, it is not difficult to see that ξ is well-defined on the cosets of Y *mod* $X = \mathrm{Ker}(g)$, precisely when it is well-defined on the 0-coset X, i.e. when it restricts to X. We shall state this property as follows:

(KC) An action passes to the quotient if, and only if, it restricts to the kernel.

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[1] The arrow labeled ξ in the diagram is wavy in order to emphasize that it is not a group homomorphism, i.e. that it is not *internal* to the category of groups.

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Action of quotients

Suppose now that we are given a group action ξ as before, and a surjective group homomorphism $q\colon A \to Q$. A natural question arises: when does the given A-action induce a Q-action? In this case, the restriction of the action ξ to the kernel K of q always exists, and the condition under which the action of the quotient is well defined, amounts to the fact that the kernel of q acts trivially.

These issues can be addressed in any category where a notion of internal object action is available, e.g. in any semi-abelian category (see [10]). Indeed, we will show that the property (KC) characterizes strongly protomodular categories among semi-abelian categories, and that, in such contexts, actions of quotients behave substantially in the same way as in the case of groups.

On the other hand these issues can be dealt with also in more general contexts. Indeed, when an object A acts on object X, just like in the case of group, one can consider the split epimorphism $X \rtimes A \to A$ given by the semidirect product projection together with its canonical section. Vice-versa, any split epimorphism with codomain A gives rise to the *conjugation* A-action on the kernel of the split epimorphism.

This allows to formulate our issues in terms of split epimorphisms, or *points*, even in contexts where the machinery of internal actions is not at all available. This line of investigation will lead us to the study of some new classifying aspects of the fibration of points. In particular, with Proposition 3.3, we will give a characterization of protomodular categories among quasi-pointed regular ones as those with kernel functors that reflect short exact sequences. Then, we will show that the problem of extending actions along quotients translates (in terms of points) in a property closely connected with strong protomodularity, i.e. the fact that kernel functors reflect kernels. In fact, this property coincides with strong protomodularity in the semi-abelian case (Proposition 3.6). On the other hand, the property of extending an action along a quotient of the acting object has a counterpart in terms of points in a property of change of base functors, as described in Proposition 4.1. This observation eventually provides an exhaustive description of change of base functors of the fibration of points along a regular epimorphism.

Our work confirms that strongly protomodular categories are a convenient setting for working with internal actions, and related constructions. Indeed, in the (strongly semi-abelian) varietal case, not only internal actions can be described externally, i.e. with suitable set-theoretical maps, but also, they behave *nicely* with respect to quotients. This fact allows to apply varietal techniques to the intrinsic setting.

Many varieties of universal algebra are strongly protomodular: the categories of groups, Lie algebras, rings and, more generally, all distributive Ω_2-groups, i.e. distributive Ω-groups with only unary and binary operations (see [11]), as for instance the categories of interest in the sense of G. Orzech [12].

The paper is organized as follows.

In Section 2 we recall the basic notions and fix the notation. Section 3 and Section 4 are quite independent to each other. Section 3 is devoted to the study of the exactness properties of kernel functors. We prove that in quasi-pointed regular categories, protomodularity is equivalent to the fact that kernel functors reflect short exact sequences. Then we give a characterization of strongly semi-abelian categories among semi-abelian ones (Theorem 5.5). In Section 4 the context is assumed to be strongly semi-abelian. Here we approach the problem of determining the conditions that make it possible to factor the change of base functor of the fibration of points along a regular epimorphism as an equivalence of categories followed by a full embedding. Actions on quotients and actions of quotients are treated explicitly in Section 5, where the results obtained in the previous sections are reconsidered in terms of internal object actions.

2. Preliminaries

Here we recall some basic notions from [4], and fix the notation.