Journal of Pure and Applied Algebra 221 (2017) 119–134

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Unobstructed Stanley–Reisner degenerations for dual quotient bundles on $G(2,n)$

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ARTICLE INFO

Article history:
Received 9 November 2015
Received in revised form 13 May 2016
Available online 1 June 2016
Communicated by R. Vakil

ABSTRACT

Let $\mathcal{Q}^*$ denote the dual of the quotient bundle on the Grassmannian $G(2,n)$. We prove that the ideal of $\mathcal{Q}^*$ in its natural embedding has initial ideal equal to the Stanley–Reisner ideal of a certain unobstructed simplicial complex. Furthermore, we show that the coordinate ring of $\mathcal{Q}^*$ has no infinitesimal deformations for $n > 5$.

1. Introduction

Let $G(2,n)$ denote the Grassmannian parametrizing 2-dimensional linear subspaces of an n-dimensional vector space, and let $I_{2,n}$ be the ideal of this variety in its Plücker embedding. Denote by $\mathcal{A}_n$ the boundary complex of the dual polytope of the n-associahedron. In [19, Proposition 3.7.4], Sturmfels shows that the Stanley–Reisner ideal associated to the join of $\mathcal{A}_n$ with an $(n-1)$-dimensional simplex is an initial ideal of $I_{2,n}$. As the boundary complex of a polytope, $\mathcal{A}_n$ has the nice property that it is pure-dimensional and topologically a sphere. Furthermore, Christophersen and the first author have shown [6, Theorem 5.6] that $\mathcal{A}_n$ is *unobstructed*, that is, the second cotangent cohomology of the associated Stanley–Reisner ring vanishes. This fact can be used to construct new degenerations of $G(2,n)$ to certain toric varieties.

In [6] it is also shown that a similar situation holds for a number of other classical varieties, including the orthogonal Grassmannian $SO(5,10)$ of isotropic 5-planes in a 10-dimensional vector space. However, no such construction is apparent for other orthogonal Grassmannians. In this paper we generalize in a different direction. The key observation is that the coordinate ring of $SO(5,10)$ in its Plücker embedding is in fact a deformation of the coordinate ring of the dual of the tautological quotient bundle on $G(2,5)$, embedded in $\mathbb{P}^9 \times \mathbb{A}^5$.

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[1] The first author was partially supported by an NSERC Discovery Grant 2015-03933.
[2] The second author was partially supported by an NSERC USRA Fellowship.

http://dx.doi.org/10.1016/j.jpaa.2016.05.029
0022-4049/

Let $\mathcal{Q}$ denote the tautological quotient bundle on $G(2,n)$, and $\mathcal{Q}^*$ its dual, see §2.3. The bundle $\mathcal{Q}^*$ comes with a natural embedding in $G(2,n) \times \mathbb{A}^n$, and hence (after composition with the Plücker embedding) an embedding in $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{A}^n$. Let J_n denote the ideal of $\mathcal{Q}^*$ in this embedding. Our first main result (Theorem 3.1) is to show that under an appropriate term order, the initial ideal of J_n is the Stanley–Reisner ideal associated to the join of a $2n-4$-dimensional simplex with a simplicial complex $\mathcal{K}_n$. This complex $\mathcal{K}_n$ is obtained from the bipyramid over $\mathcal{A}_n$ via a stellar subdivision, see §2.2 for details. It shares many of the nice properties of $\mathcal{A}_n$: topologically it is a sphere, and by a result of [4] it is also unobstructed.

Motivated by the fact that for $n = 5$, the coordinate ring of $\mathcal{Q}^*$ deforms to that of $SO(5,10)$, we consider the deformation theory of the coordinate ring S_n/J_n of $\mathcal{Q}^*$. Our second main result (Theorem 5.1) is that $T^1(S_n/J_n) = 0$ if $n > 5$, that is, S_n/J_n has no infinitesimal deformations. The proof of this fact relies heavily on our first result. Along the way, we describe the syzygies of J_n in §4, and obtain an algebraic proof of the result of Svanes [20] that the coordinate ring of $G(2,n)$ has no infinitesimal deformations for $n \geq 5$.

Our motivation for the above two results comes from our desire to better understand higher-dimensional Fano varieties, along with their degenerations to toric varieties. The bundle $\mathcal{Q}^*$ is related in a natural fashion to two projective varieties: its projectivization $\mathbb{P}(\mathcal{Q}^*)$, and the closure of $\mathcal{Q}^*$ in an appropriate projective space, see §6. Complete intersections of sufficiently low degree in both these varieties are Fano, and they come equipped with degenerations to unobstructed Stanley–Reisner schemes due to our result above on the initial ideal of J_n. This can be used as in [6] to find toric degenerations of these Fano varieties. Moreover, we are able to use our rigidity result on S_n/J_n to show that both the projectivization and the projective closure of $\mathcal{Q}^*$ are rigid, and that any class of hypersurfaces in one of these varieties is closed under deformation, see Theorem 6.5.

2. Preliminaries

2.1. Simplicial complexes and Stanley–Reisner ideals

Let V be any finite set, and $\mathcal{P}(V)$ its power set. An abstract *simplicial complex* on V is any subset $\mathcal{K} \subset \mathcal{P}(V)$ such that if $f \in \mathcal{K}$ and $g \subset f$, then $g \in \mathcal{K}$. Elements $f \in \mathcal{K}$ are called *faces*; the dimension of a face f is $\dim f := \#f - 1$. Zero-dimensional faces are called *vertices*; one-dimensional faces are called *edges*. By Δ_n we denote n-dimensional simplex, the power set of $\{0,\dots,n\}$. By Δ_{-1} we denote the empty set. By $\partial\Delta_n$ we denote the boundary of Δ_n, that is, $\partial\Delta_n = \Delta_n \setminus \{0,\dots,n\}$.

Fix an algebraically closed field $\mathbb{K}$. Let $\mathcal{K} \subset \mathcal{P}(V)$ be any simplicial complex. Its *Stanley–Reisner ideal* is the square-free monomial ideal $I_\mathcal{K} \subset \mathbb{K}[x_i \mid i \in V]$

$$I_\mathcal{K} := \langle x_p \mid p \in \mathcal{P}(V) \setminus \mathcal{K} \rangle$$

where for $p \in \mathcal{P}(V)$, $x_p := \prod_{i \in p} x_i$. This gives rise to the *Stanley–Reisner ring* $A_\mathcal{K} := \mathbb{K}[x_i \mid i \in V]/I_\mathcal{K}$. We refer to [18] for more details on Stanley–Reisner theory.

We will be interested in two operations on simplicial complexes. Given simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ on disjoint sets, their *join* is the simplicial complex

$$\mathcal{K} * \mathcal{L} = \{f \cup g \mid f \in \mathcal{K},\ g \in \mathcal{L}\}.$$

Given a simplicial complex $\mathcal{K}$ and a face $f \in \mathcal{K}$ of dimension at least one, the *stellar subdivision* of $\mathcal{K}$ at f is the simplicial complex

$$\mathrm{st}(f,\mathcal{K}) = \{g \in \mathcal{K} \mid f \nsubseteq g\} \cup \{g \cup v \in \mathcal{K} * \Delta_0 \mid f \nsubseteq g \text{ and } f \cup g \in \mathcal{K}\}$$