Journal of Pure and Applied Algebra 221 (2017) 135–143

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

The snail lemma in a pointed regular category

Zurab Janelidze [a,1], Enrico M. Vitale [b,*]

[a] *Mathematics Division, Department of Mathematical Sciences, Stellenbosch University, Private Bag X1, Matieland 7602, South Africa*
[b] *Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, 2 ch. du Cyclotron, 1348 Louvain-la-Neuve, Belgium*

ARTICLE INFO

Article history:
Received 1 January 2016
Received in revised form 1 June 2016
Available online 17 June 2016
Communicated by J. Adámek

MSC:
18G50; 18C99; 20J15; 08B05; 08C05

ABSTRACT

In this paper we explore the snail lemma in a pointed regular category. In particular, we show that under the presence of cokernels of kernels, the validity of the snail lemma is equivalent to subtractivity of the category. As a corollary, this gives that in the more restrictive context of a normal category the validity of the snail lemma is equivalent to the validity of the snake lemma.

© 2016 Elsevier B.V. All rights reserved.

1. Introduction

In studying the Hochschild–Serre exact sequence associated with a surjective homomorphism $p\colon E \to G$ of groups, and a G-module M, the second author noticed that such a sequence can be obtained by embedding p and M in the 2-category of categorical groups and then using a general form of the Brown exact sequence associated with a fibration of pointed groupoids (see [8–10]). The advantage of this unusual technique is that it becomes apparent that the Hochschild–Serre exact sequence exists even if p is not surjective, and that the surjectivity of p is only needed to identify the middle point of the sequence with the group of equivariant homomorphisms from the kernel of p to M.

This suggests to look for the *snail lemma*, a new diagram lemma similar to the classical snake lemma and having as starting point any commutative square of the form

* Corresponding author.
E-mail addresses: zurab@sun.ac.za (Z. Janelidze), enrico.vitale@uclouvain.be (E.M. Vitale).

[1] Supported by the South African National Research Foundation. The first author is grateful for the kind hospitality of Université catholique de Louvain.

http://dx.doi.org/10.1016/j.jpaa.2016.06.001
0022-4049/© 2016 Elsevier B.V. All rights reserved.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

where α and β satisfy suitable conditions but where, in contrast with the snake lemma, we do not require that f is surjective (recall that the fact that f is surjective enters in the snake lemma in order to construct the homomorphism connecting the "kernel part" and the "cokernel part" of the sequence). In [19] the second author proved that the snail lemma holds true in any pointed regular category [4] which is at the same time a protomodular category [6], and in this context naturally subsumes the snake lemma (the snake lemma was established in this context by D. Bourn in [7]). In this paper we show that

- already in a normal category [13], i.e. in a pointed regular category where every regular epimorphism is a normal epimorphism (that is, a cokernel), which is less than being protomodular, the validity of the snake lemma can be deduced from that of the snail lemma,
- and moreover, in a normal category the snail lemma is equivalent to the snake lemma, as well as to the 3×3 lemma;
- we deduce these two facts from a more general technical result that in a pointed regular category having cokernels of kernels, the snail lemma is equivalent to subtractivity of the category in the sense of [12].

Note that subtractivity is a much weaker property than protomodularity. It arises naturally in the theory of ideals and related topics in universal algebra — see [11,17,1–3,18]. At the same time, subtractivity can be seen as an implicit ingredient for diagram chasing arguments, appearing already in [16,5] (see [14]).

2. Preliminaries

In this paper we will use the technique of "chasing subobjects", which goes back to Mac Lane [15,16], and which, in the context of a pointed regular category was made explicit in [13]. Let $\mathbb{C}$ be such a category. Then there is a *pointed subobject functor* $\mathcal{S} : \mathbb{C} \to \mathbf{Set}_*$ from $\mathbb{C}$ to the category $\mathbf{Set}_*$ of pointed sets. It sends an object X of $\mathbb{C}$ to the pointed set of subobjects of X, with the base point given by the "null subobject", i.e. the subobject given by the null morphism $0 \to X$ from the null object 0 to X. This functor preserves and reflects null objects, null morphisms, and regular epimorphisms (which in $\mathbf{Set}_*$ are the same as surjective morphisms between pointed sets), and preserves monomorphisms and weak pullbacks. Moreover, the pointed subobject functor also preserves and reflects exactness of a sequence, when exactness is defined in the following "non-symmetric" way: call a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of morphisms in a pointed regular category *exact* at Y when, in any (and hence every) decomposition $f = m \cdot e$ of f into a regular epimorphism e followed by a monomorphism m, the monomorphism m is a kernel of g. The category $\mathbf{Set}_*$ is itself a pointed regular category, and in this category the above condition simply states that an element of Y is mapped by g to the base point of Z, if and only if some element from X is mapped to it by f. These properties of the pointed subobject functor allow to reduce proving diagram lemmas in an abstract pointed regular category to proving them in $\mathbf{Set}_*$ (see [13,14]). In fact, even the additional conditions on $\mathbb{C}$ which are sufficient (and which, in many cases also turn out to be necessary) to prove some of the standard diagram lemmas can also be conveniently captured by the pointed subobject functor. For example, the condition of subtractivity [12] of $\mathbb{C}$, which (as shown in [13]) is necessary

Download English Version:

https://daneshyari.com/en/article/4595731

Download Persian Version:

https://daneshyari.com/article/4595731

Daneshyari.com