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Ideals in deformation quantizations over $\mathbb{Z}/p^n\mathbb{Z}$

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A B S T R A C T

Let $\mathbf{k}$ be a perfect field of characteristic $p > 2$. Let A_1 be an Azumaya algebra over a smooth symplectic affine variety over $\mathbf{k}$. Let A_n be a deformation quantization of A_1 over $W_n(\mathbf{k})$. We prove that all $W_n(\mathbf{k})$-flat two-sided ideals of A_n are generated by central elements.

Let $\mathbf{k}$ be a perfect field of characteristic $p > 2$. For $n \geq 1$, let $W_n(\mathbf{k})$ denote the ring of length n Witt vectors over $\mathbf{k}$. Also, $W(\mathbf{k})$ will denote the ring of Witt vectors over $\mathbf{k}$. As usual, given an algebra B its center will be denoted by $Z(B)$. Throughout the paper we will fix once and for all an affine smooth symplectic variety X over $\mathbf{k}$, and an Azumaya algebra A_1 over X (equivalently over $\mathcal{O}_X$). Thus, we may (and will) identify the center of A_1 with $\mathcal{O}_X$-the structure ring of $X : Z(A_1) = \mathcal{O}_X$. Let $\{,\}$ denote the corresponding Poisson bracket on $\mathcal{O}_X$. A deformation quantization of A_1 over $W_n(\mathbf{k}), n \geq 1$ is, by definition, a flat associative $W_n(\mathbf{k})$-algebra A equipped with an isomorphism $A/pA \simeq A_1$ such that for any $a, b \in A$ such that $a \ mod \ p \in \mathcal{O}_X, b \ mod \ p \in \mathcal{O}_X$, one has

$$\{a \ mod \ p, b \ mod \ p\} = (\frac{1}{p}[a, b]) \ mod \ p.$$

One defines similarly a quantization of A_1 over $W(\mathbf{k})$.

Main result of this note is the following

Theorem 1. *Let A be a deformation quantization over $W_n(\mathbf{k})$ of an Azumaya algebra A_1 over X. Let $I \subset A$ be a two-sided ideal which is flat over $W_n(\mathbf{k})$. Then I is generated by central elements: $I = (Z(A) \cap I)A$.*

[1]Before proving this result we will need to recall some results of Stewart and Vologodsky [3] on centers of certain algebras over $W_n(\mathbf{k})$.

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[1] We showed in [6] that Hochschild cohomology of a quantization A is isomorphic to the de Rham-Witt complex $W_n\Omega^*_X$ of X.

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Throughout for an associative flat $W_n(\mathbf{k})$-algebra R, we will denote its reduction $\mod p^m$ by $R_m = R/p^mR$. Also center of an algebra R_m will be denoted by $Z_m, m \leq n$. Recall that in this setting there is the natural deformation Poisson bracket on Z_1 defined as follows. Given $z, w \in Z_1$, let $\tilde{z}, \tilde{w}$ be lifts in R of z, w respectively. Then put

$$\{z, w\} = \frac{1}{p}[\tilde{z}, \tilde{w}] \mod p.$$

In this setting, Stewart and Vologodsky [3, formula (1.3)] constructed a ring homomorphism $\phi_m : W_m(Z_1) \to Z_m$ from the ring of length m Witt vectors over Z_1 to Z_m, defined as follows

$$\phi_n(z_1, \cdots, z_m) = \sum_{i=1}^{m} p^{i-1}\tilde{z_i}^{p^{m-i}}$$

where $\tilde{z_i} \in R$ is a lift of $z_i, 1 \leq i \leq m$. We also have the following natural maps

$$r : Z_m \to Z_{m-1}, r(x) = x \mod p^{m-1}, v : Z_{m-1} \to Z_m, v(x) = p\tilde{x}$$

where $\tilde{x}$ is a lift of x in R_m. On the other hand on the level of Witt vectors of Z_1, we have the Verschibung map $V : W_m(Z_1) \to W_{m+1}(Z_1)$ and the Frobenius map $F : W_m(Z_1) \to W_{m-1}(Z_1)$. It was checked in [3] that above maps commute

$$\phi_{m-1}F = r\phi_m, \quad \phi_m V = v\phi_{m-1}.$$

We will recall the following crucial computation from [3]. Let $x = \phi_m(z), z = (z_1, \cdots, z_m) \in W_m(Z_1)$ and let $\tilde{x}$ be a lift of x in R_{m+1}. Then it was verified in [3] that the following inequality holds in $Der_{\mathbf{k}}(Z_1, Z_1)$

$$\delta_x = (\frac{1}{p^m}[\tilde{x}, -]) \mod p|_{Z_1} = \sum_{i=1}^{m} z_i^{p^{m-i}-1}\{z_i, -\} \tag{2}$$

The main result of [3, Theorem 1] states that if $\operatorname{Spec} Z_1$ is smooth variety and the deformation Poisson bracket on Z_1 is induced from a symplectic form on $\operatorname{Spec} Z_1$, then the map ϕ_m is an isomorphism for all $m \leq n$. In particular,

$$Z_1{}^{p^m} = Z_{m+1} \mod p.$$

We will need the following slight generalization of this result. Its proof follows very closely to the one in [3, Theorem 1].

Proposition 3. *Let $n \geq 1$ and $m \subset \mathcal{O}_X$ be an ideal, and let $B = \mathcal{O}_X/m^{p^n}\mathcal{O}_X$. Let R be an associative flat $W_n(\mathbf{k})$-algebra such that $Z(R/pR) = B$ and the corresponding deformation Poisson bracket on B coincides with the one induces from X. Then*

$$Z(R) = \phi_n(W_n(B)), \quad Z(R) \cap pR = \phi_n(VW_{n-1}(B)).$$

Just as in [3, Lemma 2.7] the following result plays the crucial role.

Lemma 4. *Let $z_1, \cdots, z_n \in B$ be such that $\sum_{i=1}^{n} z_i^{p^{n-i}-1}dz_i = 0$. Then $z_i \in B^p + \bar{m}^{p^i}B$, where $\bar{m} = m/m^{p^n}\mathcal{O}_X$.*