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On parabolic Kazhdan–Lusztig R-polynomials for the symmetric group

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ABSTRACT

Parabolic R-polynomials were introduced by Deodhar as parabolic analogues of ordinary R-polynomials defined by Kazhdan and Lusztig. In this paper, we are concerned with the computation of parabolic R-polynomials for the symmetric group. Let S_n be the symmetric group on $\{1, 2, \ldots, n\}$, and let $S = \{s_i \,|\, 1 \le i \le n-1\}$ be the generating set of S_n, where for $1 \le i \le n-1$, s_i is the adjacent transposition. For a subset $J \subseteq S$, let $(S_n)_J$ be the parabolic subgroup generated by J, and let $(S_n)^J$ be the set of minimal coset representatives for $S_n/(S_n)_J$. For $u \le v \in (S_n)^J$ in the Bruhat order and $x \in \{q, -1\}$, let $R^{J,x}_{u,v}(q)$ denote the parabolic R-polynomial indexed by u and v. Brenti found a formula for $R^{J,x}_{u,v}(q)$ when $J = S \setminus \{s_i\}$, and obtained an expression for $R^{J,x}_{u,v}(q)$ when $J = S\setminus\{s_{i-1}, s_i\}$. In this paper, we provide a formula for $R^{J,x}_{u,v}(q)$, where $J = S \setminus \{s_{i-2}, s_{i-1}, s_i\}$ and i appears after $i-1$ in v. It should be noted that the condition that i appears after $i-1$ in v is equivalent to that v is a permutation in $(S_n)^{S\setminus\{s_{i-2},s_i\}}$. We also pose a conjecture for $R^{J,x}_{u,v}(q)$, where $J = S \setminus \{s_k, s_{k+1}, \ldots, s_i\}$ with $1 \le k \le i \le n-1$ and v is a permutation in $(S_n)^{S\setminus\{s_k,s_i\}}$.

1. Introduction

Parabolic R-polynomials for a Coxeter group were introduced by Deodhar [4] as parabolic analogues of ordinary R-polynomials defined by Kazhdan and Lusztig [7]. In this paper, we consider the computation of parabolic R-polynomials for the symmetric group. Let S_n be the symmetric group on $\{1, 2, \ldots, n\}$, and let $S = \{s_1, s_2, \ldots, s_{n-1}\}$ be the generating set of S_n, where for $1 \le i \le n-1$, s_i is the adjacent transposition that interchanges the elements i and $i+1$. For a subset $J \subseteq S$, let $(S_n)_J$ be the parabolic subgroup generated by J, and let $(S_n)^J$ be the set of minimal coset representatives of $S_n/(S_n)_J$. Assume that u and v are two permutations in $(S_n)^J$ such that $u \le v$ in the Bruhat order. For $x \in \{q, -1\}$, let $R^{J,x}_{u,v}(q)$ denote the parabolic

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R-polynomial indexed by u and v. When $J = S \setminus \{s_i\}$, Brenti [2] found a formula for $R^{J,x}_{u,v}(q)$. Recently, Brenti [3] obtained an expression for $R^{J,x}_{u,v}(q)$ for $J = S \setminus \{s_{i-1}, s_i\}$.

In this paper, we consider the case $J = S \setminus \{s_{i-2}, s_{i-1}, s_i\}$. We introduce a statistic on pairs of permutations in $(S_n)^J$ and then we give a formula for $R^{J,x}_{u,v}(q)$, where v is restricted to a permutation in $(S_n)^{S\setminus\{s_{i-2},s_i\}}$. Notice that $v \in (S_n)^{S\setminus\{s_{i-2},s_i\}}$ is equivalent to that $v \in (S_n)^J$ and i appears after $i-1$ in v. It should be noted that there does not seem to exist an explicit formula for the case when $v \in (S_n)^J$ and i appears before $i-1$ in v.

We also conjecture a formula for $R^{J,x}_{u,v}(q)$, where $J = S \setminus \{s_k, s_{k+1}, \ldots, s_i\}$ with $1 \le k \le i \le n-1$ and $v \in (S_n)^{S\setminus\{s_k,s_i\}}$. Notice also that $v \in (S_n)^{S\setminus\{s_k,s_i\}}$ can be equivalently described as the condition that $v \in (S_n)^J$ and the elements $k+1, k+2, \ldots, i$ appear in increasing order in v. This conjecture contains Brenti's formulas and our result as special cases. When $k = 1$ and $i = n-1$, it becomes a conjecture for a formula of the ordinary R-polynomials $R_{u,v}(q)$, where v is a permutation in S_n such that $2, 3, \ldots, n-1$ appear in increasing order in v.

Let us begin with some terminology and notation. For a Coxeter group W with a generating set S, let $T = \{wsw^{-1} \mid w \in W, s \in S\}$ be the set of reflections of W. For $w \in W$, the length $\ell(w)$ of w is defined as the smallest k such that w can be written as a product of k generators in S. For $u, v \in W$, we say that $u \le v$ in the Bruhat order if there exists a sequence $t_1, t_2, \ldots, t_r$ of reflections such that $v = ut_1t_2\cdots t_r$ and $\ell(ut_1\cdots t_i) > \ell(ut_1\cdots t_{i-1})$ for $1 \le i \le r$.

For a subset $J \subseteq S$, let W_J be the parabolic subgroup generated by J, and let W^J be the set of minimal right coset representatives of W/W_J, that is,

$$W^J = \{w \in W \mid \ell(sw) > \ell(w), \text{ for all } s \in J\}. \tag{1.1}$$

We use $D_R(w)$ to denote the set of right descents of w, that is,

$$D_R(w) = \{s \in S \mid \ell(ws) < \ell(w)\}. \tag{1.2}$$

For $u, v \in W^J$, the parabolic R-polynomial $R^{J,x}_{u,v}(q)$ can be recursively determined by the following property.

Theorem 1.1 *(Deodhar [4]). Let (W, S) be a Coxeter system and J be a subset of S. Then, for each $x \in \{q, -1\}$, there is a unique family $\{R^{J,x}_{u,v}(q)\}_{u,v\in W^J}$ of polynomials with integer coefficients such that for all $u, v \in W^J$,*

(i) *if $u \not\le v$, then $R^{J,x}_{u,v}(q) = 0$;*
(ii) *if $u = v$, then $R^{J,x}_{u,v}(q) = 1$;*
(iii) *if $u < v$, then for any $s \in D_R(v)$,*

$$R^{J,x}_{u,v}(q) = \begin{cases} R^{J,x}_{us,\,vs}(q), & \text{if } s \in D_R(u), \\ qR^{J,x}_{us,\,vs}(q) + (q-1)R^{J,x}_{u,\,vs}(q), & \text{if } s \notin D_R(u) \text{ and } us \in W^J, \\ (q-1-x)R^{J,x}_{u,\,vs}(q), & \text{if } s \notin D_R(u) \text{ and } us \notin W^J. \end{cases}$$

Notice that when $J = \emptyset$, the parabolic R-polynomial $R^{J,x}_{u,v}(q)$ reduces to an ordinary R-polynomial $R_{u,v}(q)$, see, for example, Björner and Brenti [1, Chapter 5] or Humphreys [6, Chapter 7]. The parabolic R-polynomials $R^{J,x}_{u,v}(q)$ for $x = q$ and $x = -1$ satisfy the following relation, so that we only need to consider the computation for the case $x = q$.

Theorem 1.2 *(Deodhar [5, Corollary 2.2]). For $u, v \in W^J$ with $u \le v$,*

$$q^{\ell(v)-\ell(u)} R^{J,q}_{u,v}\left(\frac{1}{q}\right) = (-1)^{\ell(v)-\ell(u)} R^{J,-1}_{u,v}(q).$$