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Radicals in skew polynomial and skew Laurent polynomial rings

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ABSTRACT

We provide a general procedure for characterizing radical-like functions of skew polynomial and skew Laurent polynomial rings under grading hypotheses. In particular, we are able to completely characterize the Wedderburn and Levitzki radicals of skew polynomial and skew Laurent polynomial rings in terms of ideals in the coefficient ring. We also introduce the T-nilpotent radideals, and perform similar characterizations.

0. Introduction

Pearson and Stephenson [19] characterized the prime radical of a skew polynomial ring as $P(R[x;\sigma]) = (P(R) \cap P_\sigma(R)) + P_\sigma(R)xR[x;\sigma]$ where $P_\sigma(R)$ is the so-called *σ-prime radical* of R, which is a σ-invariant ideal of R. Similarly, Bedi and Ram [1] characterized the Jacobson radical of a skew polynomial ring as $J(R[x;\sigma]) = (J(R) \cap J_\sigma(R)) + J_\sigma(R)xR[x;\sigma]$ where $J_\sigma(R)$ is a σ-invariant ideal of R. (This notation should not be confused with that utilized in [20].) Similar formulas hold for a great number of other radicals and radical-like constructions. See, for example, the papers [7,8,18].

In this paper we find that the similarities among these formulas arise, primarily, due to the fact that these radicals preserve $\mathbb{N}$-grading (in the skew polynomial case) and $\mathbb{Z}$-grading (in the skew Laurent polynomial case). In specific situations, we can do even better, by giving element-wise characterizations of these σ-skewed radicals. This is especially true when working with nilpotence properties. In particular we completely characterize the Levitzki radical and Wedderburn radical of a skew (Laurent) polynomial ring in terms of conditions in the base ring. We also introduce new radical-like ideals connected to T-nilpotence, and similarly characterize these ideals over skew (Laurent) polynomial rings.

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Throughout, all rings are assumed to be associative with 1. We let R denote an arbitrary ring, and let σ be any automorphism of R. By $R[x;\sigma]$ we mean the skew polynomial ring over R, subject to the (left) skewing condition $xr = \sigma(r)x$ for each $r \in R$. The ring of skew Laurent polynomials will be written as $R[x, x^{-1};\sigma]$. If I is an ideal in R, we will write $I \leqslant R$. For other standard notations or definitions, we refer the reader to [15].

There is one graph theoretic tool that we will use throughout the paper, called *König's tree lemma*, which we now recall.

König's tree lemma. *If G is a connected graph with infinitely many vertices and every vertex is connected to only finitely many other vertices, then there is an infinite path with no repeated vertices.*

We will apply this result with regard to zero products. To give an example, consider the collection of *nonzero* products of the form $a_1a_2\cdots a_n$ with each a_i coming from some subset S of R. We may describe this product as the ordered n-tuple $(a_1, a_2, \dots, a_n)$. We think of these tuples as vertices of a graph, with $(a_1, a_2, \dots, a_{i-1})$ connected by an edge to $(a_1, a_2, \dots, a_i)$. In particular, this graph is connected (through the empty tuple). There is an infinite path in this graph if and only if we can form a sequence of elements $a_1, a_2, \dots$ from S with $a_1a_2\cdots a_m \neq 0$ for each $m \geqslant 1$.

There are two ways we apply König's tree lemma. First, if each a_i comes from a finite set S_i (possibly even depending on $a_1, a_2, \dots, a_{i-1}$), then each vertex is connected to only finite many others. If we further have some hypothesis on the sets S_i that prevents the existence of infinite paths, then by the contrapositive of the lemma we can conclude there is an absolute bound on the length of nonzero products from S. The second way we apply the lemma is to assume (perhaps by way of contradiction) that the products under consideration can get arbitrarily long (and are finitely connected); so we can reduce to a single infinite sequence of nonzero products.

1. Definition and notation chart

In this paper we use quite a few new definitions and notations. For the convenience of the reader, we include a brief chart which collects the notations and where they are introduced in the paper (see Table 1).

Table 1
List of definitions and notations.

Name	Notation	Place introduced	Original source
ideal function	$\mathfrak{F}$	Before Lemma 2.1	New
σ-Laurent $\mathfrak{F}$-function	$\mathfrak{F}_{\sigma,\sigma^{-1}}$	Definition 2.2	New
Jacobson radical	$\mathfrak{J}(R)$	Example 2.3	Classical
σ-Laurent Jacobson radical	$\mathfrak{J}_{\sigma,\sigma^{-1}}(R)$	Example 2.3	New
upper nilradical	$\mathfrak{N}(R)$	Example 2.4	Classical
σ-Laurent upper nilradical	$\mathfrak{N}_{\sigma,\sigma^{-1}}(R)$	Example 2.4	New
bounded nilradical	$\mathfrak{B}(R)$	Example 2.5	Classical
σ-Laurent bounded nilradical	$\mathfrak{B}_{\sigma,\sigma^{-1}}(R)$	Example 2.5	New
prime radical	$\mathfrak{P}(R)$	Example 2.6	Classical
σ-Laurent prime radical	$\mathfrak{P}_{\sigma,\sigma^{-1}}(R)$	Example 2.6	New
σ-prime radical	$P_\sigma(R)$	Example 2.6	[19]
strongly σ-prime ideal		Example 2.6	[19]
Wedderburn radical	$\mathfrak{W}(R)$	Example 2.8	Classical