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Spectral threshold dominance, Brouwer's conjecture and maximality of Laplacian energy

Christoph Helmberg [a], Vilmar Trevisan [b,*]

[a] *Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany*
[b] *Instituto de Matemática, Universidade Federal do Rio Grande do Sul, CEP 91509-900, Porto Alegre, RS, Brazil*

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A B S T R A C T

The Laplacian energy of a graph is the sum of the distances of the eigenvalues of the Laplacian matrix of the graph to the graph's average degree. The maximum Laplacian energy over all graphs on n nodes and m edges is conjectured to be attained for threshold graphs. We prove the conjecture to hold for graphs with the property that for each k there is a threshold graph on the same number of nodes and edges whose sum of the k largest Laplacian eigenvalues exceeds that of the k largest Laplacian eigenvalues of the graph. We call such graphs spectrally threshold dominated. These graphs include split graphs and cographs and spectral threshold dominance is preserved by disjoint unions and taking complements. We conjecture that all graphs are spectrally threshold dominated. This conjecture turns out to be equivalent to Brouwer's conjecture concerning a bound on the sum of the k largest Laplacian eigenvalues.

* Corresponding author.
E-mail addresses: helmberg@mathematik.tu-chemnitz.de (C. Helmberg), trevisan@mat.ufrgs.br (V. Trevisan).

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1. Introduction

Let $G = (N, E)$ be a simple graph with node set $N = \{1, \ldots, n\}$ and edge set $E \subseteq \{\{i, j\}: i, j \in N, i \neq j\}$. For brevity, we will usually write ij instead of $\{i, j\}$ for edges and put $m = |E|$. It will be convenient to assume that the nodes are numbered so that their degrees $d_i = |\{j: ij \in E\}|$ are sorted non-increasingly. Let e_i denote the i-th column of the $n \times n$ identity matrix I_n and define the positive semidefinite matrices $E_{ij} := (e_i - e_j)(e_i - e_j)^T$, then the Laplacian matrix of G is defined to be $L(G) = \sum_{ij \in E} E_{ij}$. $L(G)$ may also be written as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal degree matrix and $A(G)$ is the adjacency matrix of G. If G is clear from the context, we drop the argument and simply write L. The Laplacian is a positive semidefinite matrix with a trivial eigenvalue 0 and the vector of all ones $\mathbf{1}$ as associated eigenvector. In this paper we denote the eigenvalues of L in non-increasing order by $\lambda_1(L) \geq \cdots \geq \lambda_{n-1}(L) \geq \lambda_n(L) = 0$. As the trace of L is $2m$ there holds $\sum_{i=1}^{n} \lambda_i(L) = 2m$ and for $m > 0$ at least one eigenvalue has value greater than the average degree $2m/n$. The eigenvalues of L are the Laplacian eigenvalues of G and we may write $\lambda_i(L) = \lambda_i(G)$.

The Laplacian energy of a graph G is defined to be

$$LE(G) := \sum_{i=1}^{n} \left| \lambda_i(G) - \frac{2m}{n} \right|$$

and [2] raised the question which graphs on n nodes maximize this value.

For $i = 1, \ldots, n$ the conjugate degree $d_i^*(G) = |\{i: d_i \geq i\}|$ gives the number of nodes of G of degree at least i. Each degree sequence satisfying $d_i^* = d_i + 1$ for $i = 1, \ldots, f$ with *trace* $f = \max\{i: d_i \geq i\}$ uniquely defines a graph and these graphs form the so called threshold graphs [10]. We notice that the degree sequence d of threshold graphs is fully specified once the conjugate degrees d_i^* are given for $i \in [f]$, which is easily seen by looking at a Ferrers diagram of d (see next section for definitions). There the part strictly below the diagonal boxes is the transpose of the part above and including the diagonal. The right hand side of Fig. 2 is the Ferrers diagram of the (threshold) graph corresponding to the degree sequence $d = (4, 4, 3, 3, 2, 0, 0, 0)$. The dual degrees give the number of boxes of the columns. Threshold graphs may also be characterized combinatorially by starting with the empty graph and iteratively adding a node that is isolated or fully connected to all previous nodes.

In our context, a central property of threshold graphs T is that the conjugate degrees are exactly the eigenvalues of their Laplacian matrix, $\lambda_i(T) = d_i^*(T)$ for $i = 1, \ldots, n$ [9]. It has been conjectured that among all connected graphs on n nodes the threshold graph called pineapple with trace $\lfloor \frac{2n}{3} \rfloor$ maximizes the Laplacian energy (see [12]). Among connected threshold graphs the pineapple is indeed the maximizer; for general threshold graphs on n nodes the clique of size $\lfloor \frac{2n+1}{3} \rfloor + 1$ together with $\lfloor \frac{n-3}{3} \rfloor$ isolated vertices is a threshold graph maximizing Laplacian energy [7] and we conjecture that this graph has maximum Laplacian energy among all graphs on n nodes.