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The Entringer–Poupard matrix sequence

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A B S T R A C T

The so-called Entringer–Poupard matrices naturally occur when the distribution of the statistical pair ("last letter", "greater neighbor of maximum") is under study on the set of alternating permutations. They also provide a matrix refinement of the tangent/secant numbers. Moreover, their generating function can be explicitly derived.

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1. Introduction

The aim of this paper is to construct a well-defined sequence of matrices $(A_n = (a_n(k,\ell))_{(1\le k,\ell\le n)})$ $(n\ge 1)$ with integral entries, called the *Entringer–Poupard matrix sequence*, which provides a *matrix refinement* $\sum_{k,\ell} a_n(k,\ell) = E_n$ of the *tangent and secant numbers*, in such a way that the row and column sums $\sum_\ell a_n(k,\ell)$ and $\sum_k a_n(k,\ell)$ are themselves *Entringer* and *Poupard numbers*, respectively. The sequence (A_n) is defined by a system of *partial finite difference equations* and, moreover, the generating function for the entries $a_n(k,\ell)$ of the matrices A_n can be explicitly evaluated.

This characterization of the Entringer–Poupard matrix sequence completes the program initiated in our previous papers, where matrix refinements of the tangent and secant numbers have been found having the property that *both* row and column sums were equal to Poupard numbers as in [8] and [9], and to Entringer numbers as done in [10] and [11]. There remains to say something relevant when both Entringer and Poupard numbers are involved.

1.1. Tangent and secant numbers; Entringer and Poupard numbers

The classical Euler numbers $(E_n)_{\ge 0}$ are the (integer) coefficients in exponential series expansion of the tangent resp. the secant function, viz.

$$\tan u = \sum_{n\ge 0} \frac{u^{2n+1}}{(2n+1)!} E_{2n+1} = \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \cdots,$$

$$\sec u = \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n} = 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots,$$

see e.g. [15] (pp. 177–178) or [4] (pp. 258–259) for the expansions, sequences A000182 and A000364 of the Sloane's Encyclopedia [16] for tables and more information.

The *Entringer numbers* $E_n(k)$ $(1\le k\le n)$ are traditionally defined by a *first*-order difference equation system. See, e.g., Sloane's Encyclopedia of integers [16], where they are registered as the A008282 sequence. The *Poupard numbers* $P_n(k)$ $(1\le k\le n-1)$ are registered as the A236934 and A125053 sequences, respectively, in that Encyclopedia. A full study of those latter two sequences was made in our previous paper [7]. With Δ standing for the classical finite difference operator $\Delta E_n(k) := E_n(k+1) - E_n(k)$, their definitions can be stated as follows:

$$\Delta^2 E_n(k) + E_{n-2}(k) = 0 \quad (1\le k\le n-2)$$

with the initial conditions:

$$E_1(1) = 1; \quad E_n(1) = E_n(2) = \sum_k E_{n-1}(k) \quad (n \text{ odd} \ge 3);$$