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On the dimension of the algebra generated by two positive semi-commuting matrices

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ABSTRACT

Gerstenhaber's theorem states that the dimension of the unital algebra generated by two commuting $n \times n$ matrices is at most n. We study the analog of this question for positive matrices with a positive commutator. We show that the dimension of the unital algebra generated by the matrices is at most $\frac{n(n+1)}{2}$ and that this bound can be attained. We also consider the corresponding question if one of the matrices is a permutation or a companion matrix or both of them are idempotents. In these cases, the upper bound for the dimension can be reduced significantly. In particular, the unital algebra generated by two semi-commuting positive idempotent matrices is at most 9-dimensional. This upper bound can be attained.

1. Introduction

A classical question in linear algebra asks for the upper bound of the dimension of a commutative algebra of $n \times n$ matrices. The basic question was answered by Schur

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[17] showing that the upper bound for the dimension is $\lfloor \frac{n^2}{4} \rfloor + 1$. An important version of the above question asks for the upper bound of the dimension of a unital algebra generated by two commuting matrices. Using algebraic geometry, Gerstenhaber proved the following well-known result.

Theorem 1.1 *([9]). If $n \times n$ matrices A and B commute, then the unital algebra generated by A and B is at most n-dimensional.*

As pointed out by Guralnick [10], the above theorem follows also from the irreducibility of the variety of all pairs of commuting $n \times n$ matrices, a result which was first proved by Motzkin and Taussky [14]. All these proofs of Theorem 1.1 use algebraic geometry. Purely linear-algebraic proofs, using generalized Cayley–Hamilton theorem, were provided by Barria and Halmos [3] and by Laffey and Lazarus [13]. We note that Theorem 1.1 fails for commutative algebras generated by more than 3 elements [10], while the question whether the dimension of an algebra generated by three commuting $n \times n$ matrices is bounded by n is still open. See [11] for a recent approach to this problem, and references therein.

In the case of positive matrices it is natural to consider positivity of the commutator. The study of positive commutator of positive matrices and positive operators on Banach lattices was initiated in [4]. Such commutators have interesting properties (see [4,8,5]). For example, a positive commutator $[A, B] = AB - BA$ of positive matrices A and B is nilpotent and contained in the radical of the (Banach) algebra generated by A and B. Furthermore, if one of the matrices A and B is ideal-irreducible, then positivity of their commutator implies that they actually commute. In this paper we connect the study of positive commutators of positive matrices with Gerstenhaber's theorem. More precisely, we consider the following question.

Question 1.2. *Let A and B be positive matrices with a positive commutator $AB - BA$. What is the upper bound for the dimension of the unital algebra generated by A and B?*

We prove that, in general, this dimension is at most $\frac{n(n+1)}{2}$. Then we consider Question 1.2 for special types of matrices, where the upper bound can be reduced significantly.

The paper is organized as follows. In Section 2 we gather relevant notation, definitions and properties needed throughout the text.

In Section 3 we answer the general form of Question 1.2. We show that the upper bound for the dimension in the question is $\frac{n(n+1)}{2}$. Furthermore, if one of the matrices in the question is ideal-irreducible, then the matrices commute, so that the conclusion is in this case the same as in Gerstenhaber's theorem. We also give an example of two positive semi-commuting matrices where the upper bound $\frac{n(n+1)}{2}$ is attained.

In Section 4 we consider Question 1.2 when one of the matrices is a permutation matrix. We prove that in this case the upper bound is again n. Along the way we completely describe the vector space of matrices that intertwine two cycles of possibly different sizes, and we also determine its dimension.