Linear Algebra and its Applications 512 (2017) 202–218

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Determinants of block matrices with noncommuting blocks

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ARTICLE INFO

Article history:
Received 8 May 2015
Accepted 3 October 2016
Available online 6 October 2016
Submitted by R. Brualdi

MSC:
15A15

Keywords:
Noncommutative determinant
Block matrix

ABSTRACT

Let M be an $mn \times mn$ matrix over a commutative ring R. Divide M into $m \times m$ blocks. Assume that the blocks commute pairwise. Consider the following two procedures: (1) Evaluate the $n \times n$ determinant formula at these blocks to obtain an $m \times m$ matrix, and take the determinant again to obtain an element of R; (2) Take the $mn \times mn$ determinant of M. It is known that the two procedures give the same element of R. We prove that if only certain pairs of blocks of M commute, then the two procedures still give the same element of R, for a suitable definition of noncommutative determinants. We also derive from our result further collections of commutativity conditions that imply this equality of determinants, and we prove that our original condition is optimal under a particular constraint.

1. Introduction

Let R be a commutative ring. For a positive integer m, let $M_m(R)$ be the ring of $m \times m$ matrices with entries in R. Fix positive integers m and n, and let $M \in M_{mn}(R)$. We can view M as either an $mn \times mn$ matrix over R or an $n \times n$ *block* matrix of $m \times m$ matrices

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http://dx.doi.org/10.1016/j.laa.2016.10.004

over R. Thus we can identify $M_{mn}(R)$ with $M_n(M_m(R))$. Block matrices behave in an analogous way to usual matrices with regard to addition, subtraction, and multiplication, so that these operations can be carried out blockwise or elementwise with no difference. With determinants, however, the analogy breaks down. In general, the determinants of block matrices are not even *defined*, since $M_m(R)$ is usually not commutative. However, if all blocks of M commute, then the determinant of M over $M_m(R)$ not only is well-defined (the result will be another matrix in $M_m(R)$), but also has a close relationship with the usual determinant of M over R, as we will see later in this section.

Let $\det_R M$ denote the determinant of M as a matrix over R. In [1], pp. 546–547, Bourbaki proves the following result (alternative proofs may be found in [3] and [4]):

Theorem 1.1. *Let R be a commutative ring and let S be a* commutative *subring of $M_m(R)$. Then for any matrix $M \in M_n(S) \subset M_{mn}(R)$, we have*

$$\det_R(\det_S M) = \det_R M. \tag{1}$$

Our main result strengthens Theorem 1.1 by relaxing the hypothesis that *all pairs of blocks commute*. Specifically, our main result is the following:

Theorem 1.2. *Let R be a commutative ring and let S be a* not-necessarily-commutative *subring of $M_m(R)$. Then for any matrix $M \in M_n(S) \subset M_{mn}(R)$ such that*

$$M_{ij} \text{ commutes with } M_{kl} \text{ whenever } i \neq 1, k \neq 1, \text{and } j \neq l, \tag{2}$$

where M_{ij} is the (i,j) entry of M viewed as an $n \times n$ matrix over S, we have

$$\det_R(\mathrm{Det}_S M) = \det_R M, \tag{3}$$

where $\mathrm{Det}_S M$ is the noncommutative *determinant of M over S, defined as in Definition 2.1.*

For example, when M is a 2×2 block matrix, Theorem 1.2 states: if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $CD = DC$, then $\det_R M = \det_R(AD - BC)$. This 2×2 result appears also in [4].

Theorem 1.2 will be proved in Section 4. First, in Section 2, we establish some terminology. In Section 3 we reprove Theorem 1.1. Then, by modifying Bourbaki's proof, we prove Theorem 1.2 in Section 4. In Section 5 we use Theorem 1.2 to prove two variants of itself, and in Section 6 we apply them to classify all commutativity conditions on 2×2 matrices that suffice to imply (3). Finally, in Section 7, we prove that Theorem 1.2 is optimal under a certain constraint.