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On regular graphs with four distinct eigenvalues ☆

Xueyi Huang, Qiongxiang Huang *

College of Mathematics and Systems Science, Xinjiang University, Urumqi, Xinjiang 830046, PR China

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Let $\mathcal{G}(4,2)$ be the set of connected regular graphs with four distinct eigenvalues in which exactly two eigenvalues are simple, $\mathcal{G}(4,2,-1)$ (resp. $\mathcal{G}(4,2,0)$) the set of graphs belonging to $\mathcal{G}(4,2)$ with -1 (resp. 0) as an eigenvalue, and $\mathcal{G}(4,\geq -1)$ the set of connected regular graphs with four distinct eigenvalues and second least eigenvalue not less than -1. In this paper, we prove the non-existence of connected graphs having four distinct eigenvalues in which at least three eigenvalues are simple, and determine all the graphs in $\mathcal{G}(4,2,-1)$. As a by-product of this work, we characterize all the graphs belonging to $\mathcal{G}(4,\geq -1)$ and $\mathcal{G}(4,2,0)$, respectively, and show that all these graphs are determined by their spectra.

1. Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph on n vertices with adjacency matrix $A = A(G)$. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_t$ all the distinct eigenvalues of A with multiplicities $m_1, m_2, \ldots, m_t$ ($\sum_{i=1}^{t} m_i = n$), respectively. These eigenvalues are also called the *eigenvalues* of G. All the eigenvalues together with their multiplicities are called the

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* Corresponding author.
E-mail addresses: huangxymath@gmail.com (X. Huang), huangqx@xju.edu.cn (Q. Huang).

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spectrum of G denoted by $\mathrm{Spec}(G) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \ldots, [\lambda_t]^{m_t}\}$. If G is a connected k-regular graph, then λ_1 denotes k, and has multiplicity $m_1 = 1$.

A graph G is said to be *determined by its spectrum* (DS for short) if $G \cong H$ whenever $\mathrm{Spec}_A(G) = \mathrm{Spec}_A(H)$ for any graph H. A graph G is called *walk-regular* if for which the number of walks of length r from a given vertex x to itself (closed walks) is independent of the choice of x for all r (see [16]). Note that a walk-regular graph is always regular, but in general the converse is not true.

Throughout this paper, we denote the *neighborhood* of a vertex $v \in V(G)$ by $N_G(v)$, the complete graph on n vertices by K_n, the complete multipartite graph with s parts of sizes $n_1, \ldots, n_s$ by $K_{n_1,\ldots,n_s}$, and the graph obtained by removing a perfect matching from $K_{n,n}$ by $K_{n,n}^-$. Also, the $n \times n$ identity matrix, the $n \times 1$ all-ones vector and the $n \times n$ all-ones matrix will be denoted by I_n, $\mathbf{e}_n$ and J_n, respectively.

Connected graphs with a few eigenvalues have aroused a lot of interest in the past several decades. This problem was perhaps first raised by Doob [15]. It is well known that connected regular graphs having three distinct eigenvalues are strongly regular graphs [21], and connected regular bipartite graphs having four distinct eigenvalues are the incidence graphs of symmetric balanced incomplete block designs [2,7]. Furthermore, connected non-regular graphs with three distinct eigenvalues and least eigenvalue -2 were determined by Van Dam [9]. Very recently, Cioabă et al. in [6] (resp. [5]) determined all graphs with at most two eigenvalues (multiplicities included) not equal to ± 1 (resp. -2 or 0). De Lima et al. in [17] determined all connected non-bipartite graphs with all but two eigenvalues in the interval $[-1, 1]$. For more results on graphs with few eigenvalues, we refer the reader to [1,3,4,8,11–15,18,20].

Van Dam in [8,12] investigated the connected regular graphs with four distinct eigenvalues. He classified such graphs into three classes according to the number of integral eigenvalues (see Lemma 2.1 below). Based on Van Dam's classification and the number of simple eigenvalues, we can classify such graphs more precisely, that is, if G is a connected k-regular graph with four distinct eigenvalues, then

(1) G has at least three simple eigenvalues, or
(2) G has two simple eigenvalues:
  - (2a) G has four integral eigenvalues of which two eigenvalues are simple;
  - (2b) G has two integral eigenvalues, which are simple, and two eigenvalues of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b \in \mathbb{Z}$, $b > 0$, with the same multiplicity, or

(3) G has one simple eigenvalue, i.e., its degree k:
  - (3a) G has four integral eigenvalues;
  - (3b) G has two integral eigenvalues, and two eigenvalues of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b \in \mathbb{Z}$, $b > 0$, with the same multiplicity;
  - (3c) G has one integral eigenvalue, its degree k, and the other three have the same multiplicity $m = \frac{1}{3}(n-1)$, and $k = m$ or $k = 2m$.