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The spectral characterization of butterfly-like graphs ☆

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ABSTRACT

Let $\mathbf{a}(\mathbf{k}) = (a_1, a_2, \ldots, a_k)$ be a sequence of positive integers. A *butterfly-like graph* $W_{p^{(s)};\mathbf{a}(\mathbf{k})}$ is a graph consisting of s (≥ 1) cycle of lengths $p+1$, and k (≥ 1) paths P_{a_1+1}, P_{a_2+1}, $\ldots$, P_{a_k+1} intersecting in a single vertex. The *girth* of a graph G is the length of a shortest cycle in G. Two graphs are said to be *A-cospectral* if they have the same adjacency spectrum. For a graph G, if there does not exist another non-isomorphic graph H such that G and H share the same Laplacian (respectively, signless Laplacian) spectrum, then we say that G is $L-DS$ (respectively, $Q-DS$). In this paper, we firstly prove that no two non-isomorphic butterfly-like graphs with the same girth are A-cospectral, and then present a new upper and lower bounds for the i-th largest eigenvalue of $L(G)$ and $Q(G)$, respectively. By applying these new results, we give a positive answer to an open problem in Wen et al. (2015) [17] by proving

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that all the butterfly-like graphs $W_{2^{(s)};\mathbf{a}(\mathbf{k})}$ are both $Q-DS$ and $L-DS$.

1. Introduction

Throughout this paper, G is an undirected simple graph with $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $d_G(v)$ and $N_G(v)$ be the degree and neighbor set of vertex v in G, respectively. If $d_G(v) = 1$, then we call v a *pendent vertex*. In the sequel, we enumerate the degrees of G in non-increasing order, i.e., $d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)$, where $d_G(v_i) = d_i(G)$ for $i \in \{1, 2, ..., n\}$. Sometimes, we call a vertex with maximum degree, i.e., v_1, as a *maximum degree vertex* of G.

As usual, $K_{1,n-1}$, K_n, C_n and P_n denote the star, complete graph, cycle and path with n vertices, respectively. In what follows, let $\mathbf{a}(\mathbf{k}) = (a_1, a_2, \dots, a_k)$ and $\mathbf{b}(\mathbf{k}) = (b_1, b_2, \dots, b_k)$ be two sequences of positive integers.

Suppose that $G_i \cong C_{p+1}$ with $v_i \in V(G_i)$ and $H_j \cong P_{a_j+1}$ with $u_j \in V(H_j)$, where $i = 1, 2, \dots, s$ and $j = 1, 2, \dots, k$. As shown in Fig. 1.1, a *butterfly-like graph* $W_{p^{(s)};\mathbf{a}(\mathbf{k})}$ is a graph obtained from $G_1, G_2, \cdots, G_s$ and $H_1, H_2, \cdots, H_k$ by identifying the vertices $v_1, v_2, \cdots, v_s$ and $u_1, u_2, \cdots, u_k$ (i.e., "coalescing" these vertices into a single vertex). In particular, $W_{2^{(s)};\mathbf{a}(\mathbf{1})}$ is called a *wind-wheel graph* (see [17]).

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal degree matrix of G, respectively. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. It is easy to see that $Q(G)$ is positive semidefinite (see [13]) and hence its eigenvalues can be arranged as:

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) \geq 0.$$

Note that $L(G)$ is positive semidefinite matrix with all row sums being equal to zero. Thus, we can use

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0$$

to denote the eigenvalues of $L(G)$. For the sake of brevity, if there is no risk of confusion, we always simplify $d_i(G)$, $d_G(v)$, $N_G(v)$, $\lambda_i(G)$ and $\mu_i(G)$ as d_i, $d(v)$, $N(v)$, λ_i and μ_i, respectively.

Two graphs are said to be *A-cospectral* (respectively, *Q-cospectral*, *L-cospectral*) if they have the same adjacency (respectively, signless Laplacian, Laplacian) spectrum. Similarly, two graphs are said to be *co-degree* if they have the same degree sequences. A graph G is said to be $A-DS$ (respectively, $Q-DS$, $L-DS$) if there does not