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Unitarily invariant norm inequalities for elementary operators involving G_1 operators

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In this paper, motivated by perturbation theory of operators, we present some upper bounds for $|||f(A)Xg(B) + X|||$ in terms of $|||\,|AXB| + |X|\,|||$ and $|||f(A)Xg(B) - X|||$ in terms of $|||\,|AX| + |XB|\,|||$, where A, B are G_1 operators, $|||\cdot|||$ is a unitarily invariant norm and f, g are certain analytic functions. Further, we find some new upper bounds for the Schatten 2-norm of $f(A)X \pm Xg(B)$. Several special cases are discussed as well.

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1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a separable complex Hilbert space $\mathscr{H}$ equipped with the operator norm $\|\cdot\|$. If $\dim \mathscr{H} = n$, we can identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. If $z \in \mathbb{C}$, then we write z instead of zI, where I denotes the identity operator on $\mathscr{H}$. We write $A \geq 0$ when A is positive (positive semi-definite for matrices). For any operator A in the algebra $\mathbb{K}(\mathscr{H})$ of all compact operators, we denote by $\{s_j(A)\}$ the sequence of singular values of A, i.e. the eigenvalues $\lambda_j(|A|)$, where $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. If $\operatorname{rank} A = n$, we put $s_k(A) = 0$ for any $k > n$.

In addition to the operator norm $\|\cdot\|$, which is defined on whole of $\mathbb{B}(\mathscr{H})$, a unitarily invariant norm is a map $|||\cdot||| : \mathbb{K}(\mathscr{H}) \to [0, \infty]$ given by $|||A||| = g(s_1(A), s_2(A), \cdots)$, where g is a symmetric norming function. The set $\mathcal{C}_{|||\cdot|||} = \{A \in \mathbb{K}(\mathscr{H}) : |||A||| < \infty\}$ is a closed self-adjoint ideal $\mathcal{J}$ of $\mathbb{B}(\mathscr{H})$ containing finite rank operators. It enjoys the properties:

(i) For all $A, B \in \mathbb{B}(\mathscr{H})$ and $X \in \mathcal{J}$,

$$|||AXB||| \leq ||A|| \ |||X||| \ ||B|| \, . \tag{1.1}$$

(ii) If X is a rank one operator, then

$$|||X||| = \|X\| \, . \tag{1.2}$$

Inequality (1.1) implies that $|||UAV||| = |||A|||$ for all unitary matrices $U, V \in \mathbb{B}(\mathscr{H})$ and all $A \in \mathcal{J}$. In addition, employing the polar decomposition of $X = W|X|$ with W a partial isometry and (1.1), we have

$$|||X||| = ||| \ |X| \ ||| \, . \tag{1.3}$$

The Ky Fan norms as an example of unitarily invariant norms are defined by $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A)$ for $k = 1, 2, \ldots$. The Ky Fan dominance theorem [3, Theorem IV.2.2] states that $\|A\|_{(k)} \leq \|B\|_{(k)}$ $(k = 1, 2, \ldots)$ if and only if $|||A||| \leq |||B|||$ for all unitarily invariant norms $|||\cdot|||$; see [3,9] for more information on unitarily invariant norms. For the sake of brevity, we will not explicitly mention this norm ideal. Thus, when we consider $|||A|||$, we are assuming that A belongs to the norm ideal associated with $|||\cdot|||$. It is known that the Schatten p-norms $\|A\|_p = \left(\sum_{j=1}^{\infty} s_j^p(A)\right)^{1/p}$ are unitarily invariant for $1 \leq p < \infty$; cf. [3, Section IV]. We use the notation $A \oplus B$ for the diagonal block matrix $\operatorname{diag}(A, B)$. Its singular values are $s_1(A), s_1(B), s_2(A), s_2(B), \cdots$. It is evident that

$$\|A \oplus B\| = \max\{\|A\|, \|B\|\} \quad \text{and} \quad \|A \oplus B\|_p = (\|A\|_p^p + \|B\|_p^p)^{1/p} \, . \tag{1.4}$$