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Some properties of the spectral radius for general hypergraphs

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A B S T R A C T

In this paper, the adjacency tensor of a general hypergraph is investigated. We study the Perron–Frobenius theorem for the general hypergraphs and obtain some relevant results based on it. In particular, the techniques of weighted incidence matrix and moving edge are extended to general hypergraphs for determining the structure with the maximum spectral radius. A nearly m-uniform supertree is both connected and acyclic, in which each edge contains either $m-1$ or m vertices. To begin with, the structures obtaining the maximum spectral radius in two classes of nearly uniform supertrees are determined, where one class is with given number of edges and the other is with given number of vertices.

1. Introduction

In 2005, the concept of tensor eigenvalues and the spectra of tensors were independently introduced by Qi [10] and Lim [4]. Then, in 2012, Cooper and Dutle [3] defined the eigenvalues (and the spectrum) for the uniform hypergraph, and obtained a num-

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ber of interesting results. Since these pioneering papers, the study on the spectra of hypergraphs with their various applications has attracted extensive attentions.

Based on the operation of moving edge, Li, Shao and Qi [5] proved that the supertree with the maximal spectral radius and the maximal Laplacian spectral radius is the hyperstar. Fan et al. [2] determined the hypergraphs with the maximum spectral radius over all unicyclic hypergraphs, over linear or power unicyclic hypergraphs with given girth, over linear or power bicyclic hypergraphs, respectively. To approximate the spectral radius of uniform hypergraphs, Lu and Man [6] designed the technique of weighted incidence matrix and showed that the smallest limit point of the spectral radius of connected m-uniform hypergraphs is $\rho(m) = \sqrt[m]{4}$. Moreover, the connected hypergraphs with spectral radius less than $\sqrt[m]{4}$ are classified in [6]. Along this road, they also analyzed the hypergraphs with spectral radius at most $\sqrt[m]{2+\sqrt{5}}$ and showed that they must contain a quipus-structure [7]. In [14] and [15], Yuan et al. determined the top ten supertrees with the maximum spectral radius.

Up till now, nearly all the results are about uniform hypergraphs, where each edge consists of exactly the same number of vertices. This simplifies the notations and the conclusions, but may seem restricted. The elements in the adjacency tensor of a k-uniform hypergraph have only two choices, 0 or $\frac{1}{(k-1)!}$, while the element with a nonzero value should not contain two same indices in the subscript. In [1], Banerjee et al. defined the adjacency (Laplacian and signless Laplacian) tensor for general hypergraphs, even if non-uniform. Adopting this definition, we study the Perron–Frobenius theorem and obtain a series of results based on it, showing that this extension is really natural for the general hypergraphs and the symmetric tensors have more broad applications than we thought.

Recall that a tensor $\mathcal{A}$ with order m and dimension n over the complex field $\mathbb{C}$ is a multidimensional array

$$\mathcal{A} = (a_{i_1 i_2 \cdots i_m}), 1 \leqslant i_2, i_2, \cdots, i_m \leqslant n.$$

The tensor $\mathcal{A}$ is called symmetric if its entries are invariant under any permutation of their indices. For a vector $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{C}^n$, $\mathcal{A}x^{m-1}$ is a vector in $\mathbb{C}^n$ with its i-th component defined as

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}, \forall\, i \in [n]. \tag{1.1}$$

Let $x^{[m-1]} := (x_1^{m-1}, x_2^{m-1}, \cdots, x_n^{m-1})^T \in \mathbb{C}^n$. If $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$ has a solution $x \in \mathbb{C}^n \backslash \{0\}$, then λ is called an eigenvalue of $\mathcal{A}$ and x is an eigenvector associated with λ. In particular, if $x \in \mathbb{R}^n$ is a real eigenvector of $\mathcal{A}$, clearly the corresponding eigenvalue λ is also real. In this case, x is called an H-eigenvector and λ an *H-eigenvalue*. Furthermore, if $x \in \mathbb{R}^n_+$, we say λ is an *H^+-eigenvalue* of $\mathcal{A}$. If $x \in \mathbb{R}^n_{++}$, λ is said to be an *H^{++}-eigenvalue* of $\mathcal{A}$. And the *spectral radius* of $\mathcal{A}$ is defined as