Linear Algebra and its Applications 513 (2017) 160–181

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A semi-smooth Newton method for projection equations and linear complementarity problems with respect to the second order cone [☆]

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A R T I C L E   I N F O

Article history:
Received 5 May 2016
Accepted 7 October 2016
Available online 13 October 2016
Submitted by M. Benzi

MSC:
90C33
15A48

Keywords:
Semi-smooth system
Conic programming
Second order cone
Semi-smooth Newton method

A B S T R A C T

In this paper a special semi-smooth equation associated to the second order cone is studied. It is shown that, under mild assumptions, the semi-smooth Newton method applied to this equation is well-defined and the generated sequence is globally and Q-linearly convergent to a solution. As an application, the obtained results are used to study the linear second order cone complementarity problem, with special emphasis on the particular case of positive definite matrices. Moreover, some computational experiments designed to investigate the practical viability of the method are presented.

☆ This work was supported by CNPq (Grants 303492/2013-9, 474160/2013-0, 305158/2014-7) and FAPEG.

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http://dx.doi.org/10.1016/j.laa.2016.10.007

1. Introduction

In this paper we consider the following special semi-smooth equation in $x \in \mathbb{R}^n$ associated to the closed and convex cone $\mathcal{K} \subseteq \mathbb{R}^n$:

$$\mathrm{P}_{\mathcal{K}}(\mathrm{x}) + \mathrm{Tx} = b, \tag{1}$$

where $b \in \mathbb{R}^n$ is a constant vector, T is an $n \times n$ constant nonsingular real matrix and $\mathrm{P}_{\mathcal{K}}(\mathrm{x})$ denotes the Euclidean metric projection of a vector x onto the cone $\mathcal{K}$. The equation (1) associated to the positive orthant, $\mathcal{K} = \mathbb{R}^n_{++}$, was first studied in [6]. Additional papers dealing with (1) and its variations had appeared, for instance, in [3–5,7–9,12,15,17,24,25,32].

The purpose of the present paper is to discuss the semi-smooth Newton method to solve equation (1) associated to the *second order cone*

$$\mathcal{K} := \left\{ \mathrm{x} := (x_1, \mathrm{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \ : \ \|\mathrm{x}_2\| \leq x_1 \right\}. \tag{2}$$

It is shown that, under mild assumptions, the semi-smooth Newton method applied to this equation is well-defined and the generated sequence is globally and Q-linearly convergent to a solution. As an application, we use the obtained results to study the *linear second order cone complementarity problem (LSOCCP)*: Find $\mathrm{x} \in \mathbb{R}^n$ such that

$$\mathrm{x} \in \mathcal{K}, \qquad \mathrm{Mx} + \mathrm{q} \in \mathcal{K}, \qquad \langle \mathrm{Mx} + \mathrm{q}, \, \mathrm{x} \rangle = 0, \tag{3}$$

where $\mathrm{q} \in \mathbb{R}^n$ is a constant vector, M is an $n \times n$ constant nonsingular real matrix. Complementarity problems related to the second order cone are considered in [14,20,23]. This topic of high interest is connected to several problems and has a wide range of applications, see [21]. Since this latter survey of applications many other important connections with physics, mechanics, economics, game theory, robotics, optimization and neural networks have been found, such as the ones in [2,10,19,22,28,34,35]. If M is symmetric, then the LSOCCP (3) is the optimality condition of the *quadratic programming problem under a second order cone constraint*,

$$\begin{aligned} &\text{Minimize } \frac{1}{2}\mathrm{x}^\top \mathrm{Mx} + \mathrm{q}^\top \mathrm{x} + c \\ &\qquad\quad \mathrm{x} \in \mathcal{K} \end{aligned} \tag{4}$$

where c is a real number. Although not considered in this paper, it can be shown that any second order (in particular quadratic) conic optimization problem can be reformulated in terms of complementarity problems (in particular linear) related to the second order cone, see [27].

We show that the semi-smooth Newton method for solving problems (1), (3) and (4) has interesting features, for instance, the global and linear convergence of the generated