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On the Ihara zeta function and resistance distance-based indices

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ABSTRACT

We show that the Ihara zeta function of a graph determines a resistance distance-based invariant which is a linear combination of the Kirchhoff index, additive degree-Kirchhoff index, and multiplicative degree-Kirchhoff index.

1. Introduction

Let G be a graph of order n and size m. G may have parallel edges and/or loops. The Ihara zeta function of G is a function of complex argument defined, for $|u|$ sufficiently small, by

$$Z_G(u) = \prod_{[C]} (1 - u^{\nu([C])})^{-1},$$

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where $[C]$ runs over all the prime cycles of G and $\nu([C])$ denotes the length of $[C]$. We refer the reader to [25] for an in-depth treatment of zeta functions of graphs.

Bass [1] showed that the Ihara zeta function satisfies the following determinant formula:

$$Z_G(u)^{-1} = (1-u^2)^{m-n} \det(\mathbf{I}_n - u\mathbf{A} + u^2(\mathbf{D} - \mathbf{I}_n))$$

where $\mathbf{A}$ and $\mathbf{D}$ are the adjacency and degree matrices of G, respectively.

The zeta function of a connected graph G that is md2 (i.e. it has no pendant vertices) encodes several invariants of the graph, including its order, size, number of loops, girth, and complexity (the number of spanning trees). In addition, Z_G determines whether G is regular, bipartite, or a circuit graph and, for particular classes of graphs, determines the graph's adjacency spectrum [8,9,14,23].

Motivated by the theory of electrical networks, Klein and Randic [16] introduced a distance function on a simple connected graph G, subsequently called the *resistance distance*: the resistance distance between a pair of vertices v_i and v_j of G is the effective resistance r_{ij} between v_i and v_j, when G is regarded as an electrical network with unit resistors placed on each edge.

Using the resistance distance metric, a graph invariant called the *Kirchhoff index* was defined [5,16] as

$$Kf = \sum_{1 \le i < j \le n} r_{ij}.$$

More recently, two other resistance distance-based graph invariants were put forward: the *additive degree-Kirchhoff index* [12], defined as

$$Kf^{+} = \sum_{1 \le i < j \le n} (d_i + d_j) r_{ij}$$

and the *multiplicative degree-Kirchhoff index* [6], defined as

$$Kf^{*} = \sum_{1 \le i < j \le n} d_i d_j r_{ij}$$

where d_i and d_j are the degrees of the vertices v_i and v_j.

A small sample of recent articles about the three invariants is [3,4,7,10,15,17,20–22, 26,27].

If $\mathrm{Spec}_L(G) = \{\mu_1 = 0, \mu_2, ..., \mu_n\}$ and $\mathrm{Spec}_N(G) = \{\nu_1 = 0, \nu_2, ..., \nu_n\}$ are the Laplacian and normalized Laplacian spectra of G, respectively, then [6,13]

$$Kf = n \sum_{i=2}^{n} \frac{1}{\mu_i} \tag{1}$$