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Some remarks about acyclic and tridiagonal Birkhoff polytopes

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ABSTRACT

For a given tree T we consider the facial structure of the acyclic Birkhoff polytope $\Omega(T)$. We also determine the f-vector of the polytope Ω_n^t consisting of all tridiagonal doubly stochastic matrices of order n. Finally, we count the number of combinatorially distinct faces of Ω_n^t in each dimension.

1. Introduction

Let Ω_n denote the set of all n-square doubly stochastic matrices (matrices whose entries are nonnegative and the sum of elements in each row and in each column is equal to 1). A well-known result of Birkhoff [3] and von Neumann [15] states that Ω_n is a convex $(n-1)^2$-dimensional polytope called the *Birkhoff polytope*. This polytope is the

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convex hull of all $n \times n$ permutation matrices. An intensive investigation of combinatorial and geometric properties of Ω_n has been done in [4–6].

The face lattice of Ω_n is described in [2] as the poset of all elementary subgraphs of complete bipartite graph $K_{n,n}$ ordered by inclusion. A graph G with $2n$ vertices is an *elementary graph* if every edge of G is contained in some perfect matching of G. Elementary graphs can be also described by ear decompositions. A connected bipartite graph G has an *ear-decomposition* if

$$G = x \cup P_1 \cup P_2 \cup \cdots \cup P_k,$$

where x is an edge of G and P_i is a path connecting nodes from different color classes of $G_{i-1} = x \cup P_1 \cup \cdots \cup P_{i-1}$, for all $i = 1, \dots, k$. Note that the length of P_i is odd. A bipartite graph G is elementary if and only if each component of G has an ear decomposition, see Theorem 4.1.6 in [14]. More details about elementary graphs can be found in [2] and [14].

G. Dahl in [13] considers the *tridiagonal Birkhoff polytope*

$$\Omega_n^t = \{A \in \Omega_n : A \text{ is tridiagonal }\},$$

and shows that Ω_n^t is an $(n-1)$-dimensional polytope. The number of vertices of any face of Ω_n^t is related with the number of alternating parity sequences in [12].

For a given tree T with n vertices in [8] the acyclic Birkhoff polytope $\Omega_n(T)$ is defined as the subset of Ω_n containing all matrices whose support corresponds to some subset of the set of edges of T. Note that $\Omega_n^t = \Omega_n(P_n)$, where P_n is a path with n nodes.

2. The face lattice of $\Omega_n(T)$

A nice combinatorial description of the faces of $\Omega_n(T)$ in the terms of bicolored partitions of T is introduced in [8]. A *bicolored subgraph* of a graph G is a subgraph G' for which the set of nodes $V(G')$ is partitioned into two sets: closed nodes (these nodes are colored black) and open nodes (colored white). The faces of $\Omega_n(T)$ correspond to all partitions of T into bicolored subgraphs of the following three types:

(*i*) A closed (black) node $\bullet$.
(*ii*) An open edge $\circ$–$\circ$ that connects two open (white) nodes.
(*iii*) Any connected bicolored subgraph of T, with all endpoints closed (black), different from the previous two types.

Example 1. For the tree T on Fig. 1, $\Omega_6(T)$ contains all matrices of the form

$$\begin{bmatrix} 1-x_1 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 1-x_2 & x_2 & 0 & 0 & 0 \\ x_1 & x_2 & 1-x_1-x_2-x_3 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 1-x_4 & x_4 & 0 \\ 0 & 0 & x_3 & x_4 & 1-x_3-x_4-x_5 & x_5 \\ 0 & 0 & 0 & 0 & x_5 & 1-x_5 \end{bmatrix};$$

where $1 \geqslant x_i \geqslant 0$, $x_1 + x_2 + x_3 \leqslant 1$ and $x_3 + x_4 + x_5 \leqslant 1$.