Linear Algebra and its Applications 495 (2016) 174–189

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Maps on positive definite matrices preserving Bregman and Jensen divergences ☆

Lajos Molnár [a,b,*], József Pitrik [c], Dániel Virosztek [c,b]

[a] *Department of Analysis, Bolyai Institute, University of Szeged, H-6720 Szeged, Aradi vértanúk tere 1, Hungary*
[b] *MTA-DE "Lendület" Functional Analysis Research Group, Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P.O. Box 12, Hungary*
[c] *Institute of Mathematics, Budapest University of Technology and Economics, H-1521 Budapest, Hungary*

ARTICLE INFO

Article history:
Received 6 September 2015
Accepted 7 January 2016
Available online 29 January 2016
Submitted by P. Semrl

MSC:
47B49
15B48

Keywords:
Bregman divergence
Jensen divergence
Positive definite matrices
Preservers

ABSTRACT

In this paper we determine those bijective maps of the set of all positive definite $n \times n$ complex matrices which preserve a given Bregman divergence corresponding to a differentiable convex function that satisfies certain conditions. We cover the cases of the most important Bregman divergences and present the precise structure of the mentioned transformations. Similar results concerning Jensen divergences and their preservers are also given.

☆ The first and third authors were supported by the "Lendület" Program (LP2012-46/2012) of the Hungarian Academy of Sciences. The first author was also supported by the Hungarian Scientific Research Fund (OTKA) Reg. No. K115383. The third author was partially supported by the Hungarian Scientific Research Fund (OTKA) Reg. No. K104206 and also by the "For the Young Talents of the Nation" scholarship program (NTP-EFÖ-P-15-0481) of the Hungarian State.

* Corresponding author at: Department of Analysis, Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Hungary.

E-mail addresses: molnarl@math.u-szeged.hu (L. Molnár), pitrik@math.bme.hu (J. Pitrik), virosz@math.bme.hu (D. Virosztek).

URLs: http://www.math.unideb.hu/~molnarl/ (L. Molnár), http://www.math.bme.hu/~pitrik (J. Pitrik), http://www.math.bme.hu/~virosz (D. Virosztek).

http://dx.doi.org/10.1016/j.laa.2016.01.010

1. Introduction

In a series of papers [3,8,11,9] the first author and his coauthors described the structures of surjective maps of the positive definite cones in matrix algebras, or in operator algebras which can be considered generalized isometries meaning that they are transformations which preserve "distances" with respect to given so-called generalized distance measures. This latter notion stands for any function $d : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ on any set $\mathscr{X}$ with the mere property that for $x, y \in \mathscr{X}$ we have $d(x, y) = 0$ if and only if $x = y$. We recall that in several areas of mathematics not only metrics are used to measure nearness of points but also more general functions of this latter kind.

In [11,9] the considered generalized distance measures are of the form $d = d_{N,g}$, where $N(.)$ is a unitarily invariant norm on the underlying matrix algebra or operator algebra, $g : (0, \infty) \to \mathbb{C}$ is a continuous function with the properties

(a1) $g(y) = 0$ if and only if $y = 1$;
(a2) there exists a constant $K > 1$ such that $\left|g\left(y^2\right)\right| \geq K\left|g(y)\right|$, $y > 0$,

and the generalized distance measure $d_{N,g}$ is defined by

$$d_{N,g}(X, Y) = N\left(g\left(Y^{-1/2}XY^{-1/2}\right)\right) \tag{1}$$

for all positive invertible elements X, Y of the underlying algebra. In the mentioned papers one can see several important examples of that sort of generalized distance measures, many of them having backgrounds in the differential geometry of positive definite matrices or operators. The basic tools in describing the structure of the corresponding generalized isometries have been so-called generalized Mazur–Ulam type theorems and descriptions of certain algebraic isomorphisms (Jordan triple isomorphisms) of the positive definite cones in question.

In the present paper we determine the structures of generalized isometries with respect to other important types of generalized distance measures. Namely, here we consider Bregman divergences and Jensen divergences. These types of divergences have wide ranging applications in several areas of mathematics. For example, in the recent volume [12] on matrix information geometry 3 chapters are devoted to the study of Bregman divergences. One feature of Jensen divergences which justifies their importance is that Bregman divergences can be considered as asymptotic Jensen divergences (see Section 6.2 in [12]). We further mention that the famous Stein's loss and Umegaki's relative entropy are among the most important Bregman divergences. Our basic tool in this paper to determine the corresponding preserver transformations is, just as above, also algebraic in nature but rather different from what we have mentioned in the previous paragraph. Namely, here we use order isomorphisms.

Before presenting the results we fix the notation and terminology. In what follows $\mathbb{M}_n$ denotes the algebra of all $n \times n$ complex matrices and $\mathbb{P}_n$ stands for the positive definite