Linear Algebra and its Applications 492 (2016) 9–12

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

New simple Lie algebra of uncountable dimension

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A R T I C L E   I N F O

Article history:
Received 26 August 2015
Accepted 10 November 2015
Available online 21 November 2015
Submitted by R. Brualdi

Dedicated to Prof. Olga Macedońska by grateful former student

MSC:
17B20
17B65

Keywords:
Simple Lie algebra
Infinite dimensional Lie algebra
Infinite matrix

A B S T R A C T

We introduce a new Lie algebra of infinite $\mathbb{N}\times\mathbb{N}$ matrices which have nonzero entries only in finite number of rows. We show that its subalgebra of matrices with trace 0 is uncountably dimensional simple Lie algebra.

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1. Introduction

Let K be a field of characteristic 0 and let $M_n(K)$ denote a K-algebra of $n\times n$ matrices over K. It becomes a Lie algebra under Lie product $[A,B] = AB - BA$. We denote it by $\mathfrak{gl}_n(K)$. By $\mathfrak{sl}_n(K)$ we denote a Lie subalgebra of $\mathfrak{gl}_n(K)$ consisting of all matrices A with $\mathrm{tr}(A) = 0$. It is known that $\mathfrak{sl}_n(K)$ is a simple Lie algebra (it has no nontrivial

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http://dx.doi.org/10.1016/j.laa.2015.11.013
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ideals) of dimension n^2-1 [1]. The direct limit $\mathfrak{sl}_\infty(K)$ of algebras $\mathfrak{sl}_n(K)$ under natural embeddings $\mathfrak{sl}_n(K) \to \mathfrak{sl}_{n+1}(K)$, given by:

$$A \to \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)$$

is a simple Lie algebra of countable dimension. It can be viewed as a Lie algebra of infinite $\mathbb{N} \times \mathbb{N}$ matrices A, which have only finite number of nonzero entries and such that $\operatorname{tr}(A) = 0$. Note that the trace 'tr' is a well defined function in this case since there is finitely many nonzero entries on the main diagonal.

In this note, we describe a new Lie algebra of infinite matrices of uncountable dimension and prove its simplicity.

Let $\mathbf{K}$ be a field of characteristic 0. Denote by $M_{rf}(\infty, \mathbf{K})$ a set of infinite $\mathbb{N} \times \mathbb{N}$ matrices over $\mathbf{K}$ having only finite number of nonzero rows. We note that a matrix in $M_{rf}(\infty, \mathbf{K})$ can have infinitely many nonzero coefficients in a nonzero row. It is clear that usual addition of two matrices $A = (a_{ij}), B = (b_{ij}) \in M_{rf}(\infty, \mathbf{K})$, $A + B = (a_{ij} + b_{ij})$, is well defined. Similarly, the multiplication of a matrix A from the left by a scalar $a \in \mathbf{K}$, given by $a \cdot A = (a \cdot a_{ij})$, is also well defined. Looking at a standard matrix multiplication of two matrices $C = A \cdot B$, given by the formula $c_{ij} = \sum_{k=1}^{\infty} a_{ik} b_{kj}$ we see that in this infinite sum there is only a finite number of nonzero summands $a_{ik} b_{kj}$ and thus c_{ij} is well defined. So, $M_{rf}(\infty, \mathbf{K})$ is an associative $\mathbf{K}$-algebra.

Thus $M_{rf}(\infty, \mathbf{K})$, with respect to Lie product $[A, B] = AB - BA$, forms a Lie algebra denoted by $\mathfrak{gl}_{rf}(\infty, \mathbf{K})$. Every matrix U in $\mathfrak{gl}_{rf}(\infty, \mathbf{K})$ has only finite number of nonzero entries on the main diagonal, we can define a trace $\operatorname{tr}(U)$ as a sum of these nonzero diagonal entries. By $\mathfrak{sl}_{rf}(\infty, \mathbf{K})$ we denote a Lie subalgebra of $\mathfrak{gl}_{rf}(\infty, \mathbf{K})$ consisting of matrices U such that $\operatorname{tr}(U) = 0$.

Our main result is the following

Theorem 1.1. *$\mathfrak{sl}_{rf}(\infty, \mathbf{K})$ is a simple Lie algebra of uncountable dimension.*

For any $i, j \in \mathbb{N}$ denote by E_{ij}, the matrix unit, the infinite matrix whose only nonzero entry is 1 in the (i, j) position. Sometimes, if there is no ambiguity we denote by E_{ij} its finite $n \times n$ analogue. The product of any matrix units E_{ij} and E_{kl} is the following

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj},$$

where δ_{ij} – Kronecker's symbol. If i, j, k are pairwise distinct, then we have a known Chevalley formula

$$[E_{ik}, E_{kj}] = E_{ij}.$$

The set $\{E_{ij} \mid i, j \in \mathbb{N}\}$ forms a basis of $\mathfrak{gl}_\infty(\mathbf{K})$, and the set $\{E_{ij}, E_{rr} - E_{ss} \mid i, j, r, s \in \mathbb{N}, i \neq j, r \neq s\}$ forms a generating set for $\mathfrak{sl}_\infty(\mathbf{K})$. Thus $\mathfrak{gl}_\infty(\mathbf{K})$ and $\mathfrak{sl}_\infty(\mathbf{K})$

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