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Maximizing the spectral radius of a matrix product

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A B S T R A C T

For a non-negative matrix A the spectral radius of the product XA is maximized over all non-negative diagonal matrices X with trace 1. Instead of following the naive approach of solving a sequence of matrix eigenvalue problems, we construct a related minimization problem, with a rather simple gradient flow, and follow this flow with a steepest descent method. This procedure gives lower bounds and eventually the solution with desired accuracy. On the other hand, we obtain an upper bound in the form of the max algebra Perron root of the matrix A (and some refined upper bounds). Numerical experiments show that in many cases the upper bound is a surprisingly good estimate.

1. The problem

We consider a variational problem for a non-negative $n \times n$ matrix A. We use $e^T = (1, \dots, 1)$ and define the simplex $\Delta = \{x \in \mathbb{R}^n_+ : e^T x = 1\}$. For any vector $x = (x_i)$ let

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$X = (x_i\delta_{ij})$ be the corresponding diagonal matrix, similarly for y, Y, v, V, etc. Let ρ denote the spectral radius.

Problem. For a given non-negative matrix A find

$$\rho_{\max}(A) = \max_{x\in\Delta} \rho(XA). \tag{1}$$

Since the spectral radius is a continuous function on the space of matrices and the set Δ is compact, there is an optimal vector x for which the maximum is assumed. The vector x may not be unique.

The number $\rho_{\max}(A)$ can also be characterized as

$$\rho_{\max}(A) = \max\bigl\{\rho(VAV) : v \in \mathbb{R}^n_+,\ v^T v = 1\bigr\}. \tag{2}$$

Such a problem may show up in a context where interactions between groups are known but the group sizes are not known (worst case scenario) or can be chosen (optimal allocation).

In this section we give some examples where this problem occurs and we present some classes of matrices A for which the solution is evident. In Section 2 we prove a maximum principle for $\rho_{\max}(A)$ which gives lower bounds without computing eigenvectors. The underlying idea is to vary a vector y and then construct x in such a way that y is an eigenvector of XA. This principle is used to get improved lower bounds by a steepest descent (or rather ascent) method in Section 3. In Section 4 upper bounds for $\rho_{\max}(A)$ are obtained with tools from max algebra. We report numerical experiments in Section 5.

Now we give examples as announced before.

Example 1. Assume an economy with n sectors. Let a_{ij} be the technical flow coefficients in the sense of input–output analysis. Resources x_i (or total initial production) can be allocated to the sectors whereby the total amount is limited by $\sum_i x_i = 1$. In a linear economic model one varies the distribution $x = (x_i)$ and asks for the maximal output. In a dynamic version of a model one looks for a balanced growth solution where the output distribution is the eigenvector to the largest eigenvalue of the matrix AX; this eigenvalue is the same for the matrix XA.

The above is a “forward” version of an input–output model whereas the classical model of Leontief [10] is mostly seen as a ‘backward in time” model for a finite number of time steps. The dynamic model and balanced growth have been studied for Leontief’s model, see [12,3].

Example 2. Assume a deterministic multi-type epidemic model with n distinct groups where infection and recovery do not change the group status. The disease can be transmitted between groups. Let $B = (b_{ij})$ be the matrix of transmission rates from group j to group i, and let $D = (d_i\delta_{ij})$ be the diagonal matrix of recovery rates. Let $x = (x_i)$