Linear Algebra and its Applications 464 (2015) 90–99

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A backward error for the symmetric generalized inverse eigenvalue problem

Wei Ma [1]

School of Mathematics and Statistics, Nanyang Normal University, Nanyang, Henan 473061, People's Republic of China

ARTICLE INFO

Article history:
Received 10 July 2013
Accepted 17 November 2013
Available online 5 December 2013
Submitted by J.L. Barlow

MSC:
65F18
15A18
49J52
90C56

Keywords:
Generalized inverse eigenvalue problem
Optimal backward error
Upper bound
Lower bound

ABSTRACT

In this paper, we discuss the backward error for the symmetric generalized inverse eigenvalue problem, which extends the result of Sun (1999) [10]. The optimal backward error is defined for symmetric generalized inverse eigenvalue problem with respect to an approximate solution, and the upper and lower bounds are derived for the optimal backward error. The expressions may be useful for testing the stability of practical algorithms.

1. Introduction

Throughout this paper, we use $\mathbb{R}_n^{n\times n}$ to denote the set of $n \times n$ real nonsingular matrices, $\mathcal{S}_n$ the set of $n \times n$ real symmetric matrices and $\mathcal{D}_n$ the set of $n \times n$ real diagonal matrices. $|||\cdot|||$ denotes any unitarily invariant norm, and $\|\cdot\|_2$ the spectral norm and the Euclidean vector norm. The superscript T is for transpose. The symbol I is the $n \times n$ identity matrix. $A > 0$ ($A \geqslant 0$) denotes that A is a symmetric positive definite matrix (symmetric positive semidefinite matrix). For four matrices A, B, C and $D \in \mathcal{S}_n$, the relation $(A, B) \simeq (C, D)$ means that A, B and C, D have the same generalized eigenvalue. $A \geqslant B$ means that $A - B$ is positive semidefinite matrix.

E-mail address: mawei7555659@126.com.

[1] The research of this author was partially supported by the Special Project Grant of Nanyang Normal University.

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.11.025

Consider the following symmetric generalized inverse eigenvalue problem:

Problem GIEP. Given $2n+2$ real symmetric $n \times n$ matrices $\{A_i\}_{i=0}^n$, $\{B_i\}_{i=0}^n$ and n real numbers $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, find $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ with real components such that

$$\big(A(\mathbf{c}), B(\mathbf{c})\big) \simeq (\Lambda, I),$$

where

$$A(\mathbf{c}) := A_0 + \sum_{i=1}^n c_i A_i, \qquad B(\mathbf{c}) := B_0 + \sum_{i=1}^n c_i B_i \geqslant \alpha I,$$

$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\alpha > 0$.

Generalized inverse eigenvalue problems arise from a remarkable variety of applications, including the discrete analogue of inverse Sturm–Liouville problems [3] and structural design [4]. A special case of the GIEP in which $B(\mathbf{c}) = I_n$ is known as the algebraic inverse eigenvalue problem [3,4]. One may refer to the survey paper [5] and [6] for the different applications and numerical methods of GIEPs. Sun [10] defined the backward error of the symmetric matrix inverse eigenvalue problems and derived an explicit expression for it. Liu and Bai [9] extended Sun's approach to more general inverse eigenvalue problems and derived the upper and lower bounds for the optimal backward error. Chen [2] extended Sun's approach to the inverse singular value problem and derived an explicit expression for the optimal backward error. However, the optimal backward error of the symmetric generalized inverse eigenvalue problem is rarely treated in the literature. In this paper we consider the optimal backward error of the symmetric generalized inverse eigenvalue problem and the upper and lower bounds are derived for the optimal backward error, which extend the result of Sun [10].

Let $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)$ be an approximate solution to Problem GIEP. In general, there are many backward perturbations $\Delta A_0, \Delta A_1, \ldots, \Delta A_n, \Delta B_0, \Delta B_1, \ldots, \Delta B_n \in \mathcal{S}_n$, and $\Delta\Lambda \in \mathcal{D}_n$ such that

$$\left(A_0 + \Delta A_0 + \sum_{i=1}^n \tilde{c}_i (A_i + \Delta A_i), B_0 + \Delta B_0 + \sum_{i=1}^n \tilde{c}_i (B_i + \Delta B_i)\right) \simeq (\Lambda + \Delta\Lambda, I),$$

and

$$B_0 + \Delta B_0 + \sum_{i=1}^n \tilde{c}_i (B_i + \Delta B_i) \geqslant \alpha I.$$

One may well ask: How close is the nearest Problem GIEP for which $\tilde{c}$ is the solution?

In order to define backward errors for measuring the distance between the original problem and the perturbed problems, we define the normwise backward error $\eta(\tilde{\mathbf{c}})$ by

$$\begin{aligned}\eta(\tilde{\mathbf{c}}) = \min_{(\Delta A_0, \Delta A_1, \ldots, \Delta A_n, \Delta B_0, \Delta B_1, \ldots, \Delta B_n, \Delta\Lambda) \in \mathcal{G}} & \left[\sum_{k=0}^n \left(\frac{|||\Delta A_k|||}{|||A_k|||} \right)^2 \right. \\ & \left. + \sum_{k=0}^n \left(\frac{|||\Delta B_k|||}{|||B_k|||} \right)^2 + \left(\frac{|||\Delta\Lambda|||}{|||\Lambda|||} \right)^2 \right]^{\frac{1}{2}} \end{aligned} \tag{1}$$

where

$$\mathcal{G} = \left\{ \begin{array}{l} (\Delta A_0, \Delta A_1, \ldots, \Delta A_n, \Delta B_0, \Delta B_1, \ldots, \Delta B_n, \Delta\Lambda): \\ (A_0 + \Delta A_0 + \sum_{k=1}^n \tilde{c}_k (A_k + \Delta A_k), B_0 + \Delta B_0 + \sum_{k=1}^n \tilde{c}_k (B_k + \Delta B_k)) \simeq (\Lambda + \Delta\Lambda, I) \\ \Delta A_0, \Delta A_1, \ldots, \Delta A_n, \Delta B_0, \Delta B_1, \ldots, \Delta B_n \in \mathcal{S}_n, \ \Delta\Lambda \in \mathcal{D}_n \end{array} \right\}. \tag{2}$$