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# On a conjecture concerning integral real roots of certain cubic polynomials ☆

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ABSTRACT

In this paper, we determine all the rational pairs $(x, n)$ such that

$$f_n(x) = 3x^3 + (7 - 10n)x^2 + 2\left(6n^2 - 11n + 8\right)x - \left(4n^3 - 6n^2 - 10n + 24\right) = 0.$$

It follows that for each positive integer $n \geqslant 5$, there is no integer solution $x$ for the polynomial. This confirms a conjecture of Li et al. concerning the uniqueness of a class of optimal graphs in the study of an extremal graph theory problem.



## 1. Introduction

In [1], Li, Tam and Su considered the problem of maximizing (also, minimizing) the absolute values of the signless Laplacian coefficients among all unicyclic graphs of a given order. They found that optimal graph for the minimization problem is unique. Believing that the optimal graph for the maximization problem is also unique, they posed the following conjecture:

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**Conjecture 1.** *For every positive integer $n \geqslant 5$, the unique real root of the cubic polynomial*

$$f_n(x) = 3x^3 + (7 - 10n)x^2 + 2\bigl(6n^2 - 11n + 8\bigr)x - \bigl(4n^3 - 6n^2 - 10n + 24\bigr)$$

*is never an integer.*

In this paper, we prove the following.

**Theorem 1.** *The rational solutions $(x, n)$ of the equation $f_n(x) = 0$ are*:

$$(2, 2), (3, 3), (4, 4), (2, 3), (3, 4), (8/3, 3), (2, 5/2), (3, 7/2).$$

By the above theorem, Conjecture 1 is true. Thus, there is a unique optimal graph for the maximization problem on unicyclic graphs.

## 2. Proof of Theorem 1

For notational convenience, we let

$$g(x, y) := f_y(x) = 3x^3 + (7 - 10y)x^2 + 2\bigl(6y^2 - 11y + 8\bigr)x - \bigl(4y^3 - 6y^2 - 10y + 24\bigr).$$

We begin with the following.

**Lemma 1.** *The finite rational points of the elliptic curve $E$ with equation $Y^2 = X(X - 1)(X + 3)$ are*:

$$(0, 0), (1, 0), (-3, 0), (-1, 2), (-1, -2), (3, 6), (3, -6).$$

**Proof.** It is well known [2] that the set of all the rational solutions of the equation together with an infinite point $\mathcal{O}$ forms the Mordell–Weil group $E(\mathbb{Q})$ of $E$, substituting $X$ by $X_1 - 1$ we will obtain the minimal Weierstrass equation of $E$:

$$E'\colon \quad Y^2 = X_1^3 - X_1^2 - 4X_1 + 4,$$

the computation by online database LMFDB [3] shows that the Mordell–Weil group $E'(\mathbb{Q})$ of $E'$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$, and we have

$$E'(\mathbb{Q}) = \bigl\{(1, 0), (2, 0), (-2, 0), (0, 2), (0, -2), (4, 6), (4, -6), \mathcal{O}\bigr\},$$

hence the finite rational points of $E$ are exactly those listed in the lemma. □

**Proof of Theorem 1.** Let $x = S + 3$, $y = S + T + 3$. By a direct computation one can show that

$$g(x, y) = S^3 + 2TS^2 + (2T - 1)S - 2T(T - 1)(2T - 1).$$