Linear Algebra and its Applications 433 (2010) 2010–2017

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Richardson method and totally nonnegative linear systems ☆

J.M. Carnicer [a], J. Delgado [b], J.M. Peña [a,*]

[a] *Department of Applied Mathematics/IUMA, University of Zaragoza, Edificio de Matemáticas, 50006 Zaragoza, Spain*
[b] *Department of Applied Mathematics/IUMA, University of Zaragoza, Escuela Universitaria Politécnica de Teruel, 44003 Teruel, Spain*

ARTICLE INFO

Article history:
Received 12 January 2010
Accepted 12 July 2010
Available online 17 August 2010

Submitted by S. Fallat

AMS classification:
15A48
15A51
65F10

Keywords:
Richardson method
Totally nonnegative matrices
Stochastic matrices

ABSTRACT

We show that modified Richardson method converges for any nonsingular totally nonnegative stochastic matrix for any choice of the parameter between 0 and 2. We present a variant of the modified Richardson method that is convergent for any nonsingular totally nonnegative matrix. We obtain the optimal parameter value for this method and give a procedure for estimating it. Numerical experiments are presented.

1. Introduction

Totally nonnegative linear systems arise commonly in many problems from approximation theory, computer-aided geometric design (CAGD), differential equations and statistics, among other fields [1,6,10].

In CAGD, the control points of an interpolating curve are the solution of a linear system whose coefficient matrix is called the collocation matrix. The collocation matrices of all shape preserving representations [2,3,10] are totally nonnegative. The progressive iteration approximation (PIA)

☆ Research Partially Supported by the Spanish Research Grant MTM2009-07315, by Gobierno de Aragón and Fondo Social Europeo.

* Corresponding author.
E-mail addresses: carnicer@unizar.es (J.M. Carnicer), jorgedel@unizar.es (J. Delgado), jmpena@unizar.es (J.M. Peña).

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doi:10.1016/j.laa.2010.07.007

property provides an iterative method for computing the control points of an interpolating curve, which is convergent if the collocation matrix is totally nonnegative [5,7]. The PIA iterations can be regarded as a particular application of Richardson's iteration for the corresponding linear system. This has motivated us to study the behaviour of Richardson's iteration for totally nonnegative matrices.

Richardson's iteration is a classical iterative method for solving linear systems whose coefficient matrices present certain spectral properties. In fact, we show that a variant of the modified Richardson method is convergent for any nonsingular totally nonnegative matrix and we find the values of the parameter for optimal convergence rate. Although the method can be slow for ill-conditioned matrices, it reveals to be a powerful tool for approximation problems by means of spline functions (interpolation, least squares, smoothing splines), which can be understood as another consequence of the optimal properties of the B-spline basis of polynomial spline spaces (cf. [3,11]).

The layout of the paper is as follows. Section 2 is devoted to the case of nonsingular stochastic totally nonnegative matrices. In Section 3, we extend the analysis to general nonsingular totally nonnegative matrices, by scaling the matrix and suggest an estimation of the optimal parameter. Finally, Section 4 includes numerical experiments.

2. Richardson method for stochastic totally nonnegative matrices

Let A be a nonsingular matrix. Richardson iterative method for solving the linear system $Ax = b$ can be written by the recurrence

$$x_{m+1} = x_m - Ax_m + b, \quad m = 0, 1, 2, \ldots \tag{1}$$

Denoting by y the new iteration obtained from x, the method can be expressed by the equation

$$y = (I - A)x + b, \tag{2}$$

where I denotes the identity matrix.

Let us recall that a matrix is *totally nonnegative* (resp., *totally positive*) if all its minors are nonnegative (resp. *positive*). A crucial property of totally nonnegative matrices can be found in Corollary 6.6 of [1].

Theorem 2.1. *All the eigenvalues of a totally nonnegative matrix are nonnegative real numbers.*

A nonnegative matrix is said to be *stochastic* if the sum of the entries of each row equals 1. Totally nonnegative stochastic matrices play a crucial role in computer-aided geometric design (see Chapter 3 of [10]). The solution of interpolation problems by curves represented in terms of a shape preserving basis leads to linear systems $Ax = b$, where A is nonsingular totally nonnegative stochastic.

Richardson iterative method always converges for nonsingular totally nonnegative stochastic matrices as the following result shows. As usual, we shall denote by $\rho(B)$ the spectral radius of a matrix B. The maximal and minimal eigenvalues of a matrix B with real spectrum will be denoted by $\lambda_{\max}(B)$ and $\lambda_{\min}(B)$, respectively.

Theorem 2.2. *Let A be a nonsingular totally nonnegative stochastic matrix. Then, the Richardson iterative method (1) converges to the solution of the system $Ax = b$ and the convergence speed corresponds to*

$$\rho(I - A) = 1 - \lambda_{\min}(A).$$

Proof. It is well-known that Richardson iterative method converges if and only if the spectral radius $\rho(I - A)$ of $I - A$ is less than 1. By Theorem 2.1, the eigenvalues of totally nonnegative matrices are nonnegative. Furthermore the spectrum is contained in (0, 1] because A is a nonsingular stochastic matrix. Clearly, the eigenvalues of $I - A$ are $1 - \lambda$ where λ is an eigenvalue of A. So the spectrum of $I - A$ is contained in [0, 1) and therefore $\rho(I - A) < 1$. □

In order to accelerate the convergence of the method, it is usual to perform a relaxation of the method, replacing the role of x_{m+1} by $(1 - w)x_m + wx_{m+1}$. In the case of Richardson method, this leads to the modified Richardson method