Appl. Comput. Harmon. Anal. 40 (2016) 352–385

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

From compression to compressed sensing

Shirin Jalali [a,*], Arian Maleki [b,*]

[a] *Department of Electrical Engineering, Princeton University, Princeton, NJ, USA*
[b] *Department of Statistics, Columbia University, New York, NY, USA*

ARTICLE INFO

Article history:
Received 13 January 2014
Received in revised form 16 October 2014
Accepted 9 March 2015
Available online 12 March 2015
Communicated by Michael Elad

Keywords:
Compression
Compressed sensing
Rate–distortion
Compressible signal pursuit
Low-complexity signals
Ito integral
Stable signal recovery

ABSTRACT

Can compression algorithms be employed for recovering signals from their underdetermined set of linear measurements? Addressing this question is the first step towards applying compression algorithms for compressed sensing (CS). In this paper, we consider a family of compression algorithms $\mathcal{C}_r$, parametrized by rate r, for a compact class of signals $\mathcal{Q} \subset \mathbb{R}^n$. The set of natural images and JPEG at different rates are examples of $\mathcal{Q}$ and $\mathcal{C}_r$, respectively. We establish a connection between the rate–distortion performance of $\mathcal{C}_r$, and the number of linear measurements required for successful recovery in CS. We then propose compressible signal pursuit (CSP) algorithm and prove that, with high probability, it accurately and robustly recovers signals from an underdetermined set of linear measurements. We also explore the performance of CSP in the recovery of infinite dimensional signals.

1. Introduction

The field of compressed sensing (CS) was established on a keen observation that if a signal is sparse in a certain basis it can be recovered from far fewer random linear measurements than its ambient dimension [1–5]. In the last decade, CS recovery algorithms have evolved to capture more complicated signal structures such as group sparsity, low-rankness [6–28], and other broader notions of "structure" [29–32]. In this paper, we consider a different type of structure based on compression algorithms. Suppose that a class of signals can be "efficiently" compressed by a compression algorithm. Intuitively speaking, such classes of signals have a certain "structure" that enables the compression algorithm to represent them with fewer bits. These structures are often much more complicated than sparsity, and employing them in CS can potentially reduce the number of measurements required for signal recovery.

* Corresponding authors.
E-mail addresses: jalali.shirin@gmail.com (S. Jalali), arian@stat.columbia.edu (A. Maleki).
URLs: https://sites.google.com/site/shirinjalali/ (S. Jalali), https://sites.google.com/site/malekiarian/ (A. Maleki).

http://dx.doi.org/10.1016/j.acha.2015.03.003

In this paper, we aim to address the following problem. Is it possible to employ compression algorithms in CS and design compression-based CS algorithms that recover signals from their under-determined set of linear measurements? As we will prove in this paper, the answer to this question is affirmative. We propose a CS-recovery algorithm based on exhaustive search over the set of "compressible" signals, that, under a certain condition on the rate–distortion performance of the code, recovers signals from fewer measurements than their ambient dimensions. Since the signal lies in the analog domain, the compression code is lossy by default, and therefore the recovery is also not perfect. The error induced by the devised recovery algorithm depends on the quality of compression code. This result provides the first theoretical basis for using generic compression algorithms in CS.

We also extend our results to analog signals. Such extensions are important for many applications including spectrum sensing [33–35]. Our generalization employs a new measurement technique, that is based on the projection of the signal onto several independent Wiener processes, and the CSP algorithm. We show that these two ingredients enable us to utilize most of our proof techniques in the problem of analog-CS as well.

The theoretical framework we develop shows a connection between the problem of compressed sensing and relevant problems in the filed of embedology [36–39]. As an application of this connection, we will derive several fundamental results of embedology as corollaries of our main results.

In this work we explore the application of a given compression code in designing a CS-recovery algorithm. In a series of related papers, the authors of [31,32,40] theoretically prove the existence of *universal* CS-recovery algorithms that require no prior knowledge about the structure of the signal. While some of the mathematical tools and techniques used here and in those papers are similar, the papers address different questions. Unlike in [32], here we are interested in designing efficient CS-recovery algorithms that are designed for a *specific* class of signals. This goal is achieved by employing a given compression code tailored for that class.

The organization of the paper is as follows. Section 2 reviews the main concepts used in this paper. Section 3 formally states the problem addressed in the paper and introduces our compressible signal pursuit algorithm. Section 4 establishes a lower bound for the number of measurements any recovery method (based on compression algorithm) requires for accurate recovery. Section 5 and Section 6 summarize our main contributions. Section 7 extends our results to the class of analog signals. Section 8 reviews the related work in the literature. Section 9 includes the proofs of our main theorems. Finally, Section 10 concludes the paper.

2. Background

2.1. Notation

Boldfaced letters such as $\mathbf{x}$ and $\mathbf{X}$ represent vectors. Calligraphic letters denote sets and operators; the distinction will be clear from the context. Given a finite set $\mathcal{A}$, $|\mathcal{A}|$ denotes its size. The ℓ_p-norm of $\mathbf{x} \in \mathbb{R}^n$ is defined as $\|\mathbf{x}\|_p \triangleq (\sum_{i=1}^n |x_i|^p)^{1/p}$. The ℓ_0-norm is also defined as $\|\mathbf{x}\|_0 \triangleq |\{i : x_i \neq 0\}|$. Note that for $p < 1$, $\|\cdot\|_p$ is a semi-norm since it does not satisfy the triangle inequality. Throughout the paper log denotes logarithm in base e, and logarithm in base 2 is denoted explicitly as $\log_2$.

2.2. Compression

Let $\mathcal{Q}$ denote a compact subset of $\mathbb{R}^n$.[1] Consider a compression algorithm for $\mathcal{Q}$ described by encoder and decoder mappings $(\mathcal{E}, \mathcal{D})$. Encoder

[1] We extend our results to infinite dimensional spaces in Section 7.