Differential Geometry and its Applications 50 (2017) 1–10

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

# Lie derivatives and structure Jacobi operator on real hypersurfaces in complex projective spaces

Juan de Dios Pérez

*Departamento de Geometria y Topologia, Universidad de Granada, 18071 Granada, Spain*

ARTICLE INFO

*Article history:*
Received 31 March 2015
Received in revised form 11 July 2016
Available online xxxx
Communicated by D.V. Alekseevsky

*MSC:*
53C15
53B25

*Keywords:*
$k$-th g-Tanaka–Webster connection
Levi-Civita connection
Complex projective space
Real hypersurface
Lie derivatives
Structure Jacobi operator

ABSTRACT

On a real hypersurface $M$ in a complex projective space we can consider the Levi-Civita connection and for any nonnull constant $k$ the $k$-th g-Tanaka–Webster connection. Associated to g-Tanaka–Webster connection we can define a differential operator of first order. We classify real hypersurfaces such that both the Lie derivative and this differential operator, either in the direction of the structure vector field $\xi$ or in any direction of the maximal holomorphic distribution coincide when we apply them to the structure Jacobi operator of $M$.



## 1. Introduction

Let $\mathbb{C}P^m$, $m \geq 2$, be a *complex projective space* endowed with the Fubini-Study metric $g$ of constant holomorphic sectional curvature 4. Let $M$ be a *connected real hypersurface* of $\mathbb{C}P^m$ without boundary. Let $\nabla$ be the Levi-Civita connection on $M$ and $J$ the complex structure of $\mathbb{C}P^m$. Take a locally defined unit normal vector field $N$ on $M$ and denote by $\xi = -JN$. This is a tangent vector field to $M$ called the structure vector field on $M$. On $M$ there exists an almost contact metric structure $(\phi, \xi, \eta, g)$, see [1], induced by the Kaehlerian structure of $\mathbb{C}P^m$, where $\phi$ is the tangent component of $J$ and $\eta$ is an one-form given by $\eta(X) = g(X, \xi)$ for any $X$ tangent to $M$. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [11–13]. His classification contains 6 types of real hypersurfaces. Kimura, [5], also proved that they are the unique Hopf real hypersurfaces with constant principal curvatures, where a real hypersurface $M$ is called Hopf if the structure vector field is principal, that is, $A\xi = \alpha\xi$ for a certain

*E-mail address:* jdperez@ugr.es.

http://dx.doi.org/10.1016/j.difgeo.2016.10.004
0926-2245/

function $\alpha$ on $M$. We will also denote by $\mathbb{D}$ the maximal holomorphic distribution on $M$, given by all vector fields orthogonal to $\xi$. Among the real hypersurfaces appearing in Takagi's list we mention the following ones:

Type $(A_1)$ real hypersurfaces are geodesic hyperspheres of radius $r$, $0 < r < \frac{\pi}{2}$. They have 2 distinct constant principal curvatures, $2\cot 2r$ with eigenspace $\mathbb{R}[\xi]$ and $\cot r$ with eigenspace $\mathbb{D}$.

Type $(A_2)$ are tubes of radius $r$, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$. They have 3 distinct constant principal curvatures, $2\cot 2r$ with eigenspace $\mathbb{R}[\xi]$, $\cot r$ and $-\tan r$. The corresponding eigenspaces of $\cot r$ and $-\tan r$ are complementary and $\phi$-invariant distributions in $\mathbb{D}$.

From now on we will call type $(A)$ real hypersurfaces to both of either type $(A_1)$ or type $(A_2)$.

Ruled real hypersurfaces in $\mathbb{C}P^m$ are characterized for the fact that $g(A\mathbb{D}, \mathbb{D}) = 0$ (see [6,7]).

The Tanaka–Webster connection, [14,16], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold. As a generalization of this connection, Tanno, [15], defined the generalized Tanaka–Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y \tag{1.1}$$

where $\nabla$ denotes the Levi-Civita connection on $M$.

Using the naturally extended affine connection of Tanno's generalized Tanaka–Webster connection, Cho defined the $k$-th g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface $M$ in $\mathbb{C}P^m$ given, see [2,3], by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.2}$$

for any X, Y tangent to $M$ where $k$ is a non-zero real number. Then $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the $k$-th g-Tanaka–Webster connection coincides with the Tanaka–Webster connection. From (1.2) the $k$-th Cho operator on $M$ associated to a tangent vector field $X$ is defined by $F^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any $Y$ tangent to $M$. Notice that if $X \in \mathbb{D}$, $F^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX$ does not depend on $k$. In this case we simply call it $F_X$. The torsion of the $k$-th g-Tanaka–Webster connection is then given by $\hat{T}^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$. Let us call $\hat{T}^{(k)}_X$ to the operator on $M$ given by $\hat{T}^{(k)}_X Y = \hat{T}^{(k)}(X, Y)$, for any $X, Y$ tangent to $M$.

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, \tilde{g})$ satisfy a very well-known differential equation. This classical differential equation naturally inspires the so-called *Jacobi operator*. That is, if $\tilde{R}$ is the curvature operator of $\tilde{M}$, and $X$ is any tangent vector field to $\tilde{M}$, the Jacobi operator (with respect to $X$) at $p \in M$, $\tilde{R}_X \in \mathrm{End}(T_p\tilde{M})$, is defined as $(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p)$ for all $Y \in T_p\tilde{M}$, being a selfadjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Clearly, each tangent vector field $X$ to $\tilde{M}$ provides a Jacobi operator with respect to $X$.

If $M$ is a real hypersurface of $\mathbb{C}P^m$ the Jacobi operator associated to the structure vector field $\xi$, $R_\xi$, is called the structure Jacobi operator on $M$.

Let $\mathcal{L}$ denote the Lie derivative on $M$. Therefore $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$ for any $X, Y$ tangent to $M$.

Pérez, Santos and Suh, [10], proved the following

**Theorem.** *Let $M$ be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, such that the structure Jacobi operator $R_\xi$ satisfies $\mathcal{L}_\xi R_\xi = 0$. Then either $M$ is locally congruent to a tube of radius $\frac{\pi}{4}$ over a complex submanifold in $\mathbb{C}P^m$ or to either a geodesic hypersphere or a tube over a totally geodesic $\mathbb{C}P^k$, $0 < k < m-1$, with radius $r \neq \frac{\pi}{4}$.*

In [4] Jeong, Pak and Suh introduced a differential operator of first order associated to the $k$-th g-Tanaka–Webster connection for a real hypersurface in a complex two-plane Grassmannian by $\mathcal{L}^{(k)}_X Y =$