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Locally conformal symplectic blow-ups [☆]

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ABSTRACT

In this paper, we study the blow-up of a locally conformal symplectic manifold. We show that there exists a locally conformal symplectic structure on the blow-up of a locally conformal symplectic manifold along a compact induced symplectic submanifold.

1. Introduction

Let M be a smooth manifold. A symplectic form on M is a 2-form $\omega \in \Omega^2(M)$ satisfying: (1) $d\omega = 0$ and (2) ω is non-degenerate, i.e. for each $p \in M$ the map

$$T_pM \ni v \longmapsto \omega(v, -) \in T_p^*M$$

is an isomorphism. It is of importance to point out that the existence of the symplectic form ω on M determines pieces of topological data: the de Rham cohomology of M with even degrees are non-vanishing and the dimension of M is even, denoted by $2n$, and there exists a homotopy class of reductions of the

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structural group of the tangent bundle TM to $\mathrm{U}(n) \simeq \mathrm{Sp}(2n;\mathbb{R})$. In particular, if M is a complex manifold and ω is the Kähler form of a Hermitian metric on M then we say that (M, ω) is a Kähler manifold.

In a more general setting, a subclass of almost symplectic manifolds called locally conformal symplectic manifolds (LCS for short) was introduced and studied by Lee [7], Liebermann[8] and Vaisman [13,14]. Intuitively, a locally conformal symplectic form is a non-degenerate 2-form ω which is conformally equivalent to a symplectic form locally. From a conformal point of view, locally conformal symplectic manifolds can be thought of the closest to symplectic manifolds. In particular, the locally conformal symplectic manifolds can serve as natural phase spaces of Hamiltonian dynamical systems and from the geometric aspect it appears in the study of contact manifolds and Jacobi manifolds (cf. [1,6,14]). Likewise, if M is a complex manifold and the locally conformal symplectic form ω on M is the Kähler form of a Hermitian metric h then we say that (M, ω) is a locally conformal Kähler manifold (LCK for short) (cf. [5]). To make this more precisely, we have the following diagram explaining the relationships between symplectic/Kähler manifolds and locally conformal symplectic/Kähler manifolds:

$$\begin{array}{ccc} \{\text{Kähler manifolds}\} & \subset & \{\text{LCK manifolds}\} \\ \cap & & \cap \\ \{\text{Symplectic manifolds}\} & \subset & \{\text{LCS manifolds}\}. \end{array}$$

It is well known that the blow-up is a very useful operation in symplectic/Kähler geometry. In particular, the Kähler property is preserved under blow-ups. In the symplectic category, it was McDuff [9] who first proved that the blow-up of a symplectic manifold along a compact symplectic submanifold also admits a symplectic structure, moreover, using this symplectic blow-up technique she constructed the first simply-connected, symplectic manifold which is non-Kähler. For locally conformal Kähler manifolds, Tricerri [12] and Vuletescu [15] proved that the blow-up of a locally conformal Kähler manifold at a point has a locally conformal Kähler structure. In 2013, using the current theory on locally conformal Kähler manifolds, Ornea–Verbitsky–Vuletescu [11] showed that the blow-up of a locally conformal Kähler manifold along a submanifold is locally conformal Kähler if and only if the submanifold is globally conformally equivalent to a Kähler submanifold. In the locally conformal symplectic case, Y. Chen and the first named author [4] introduced the definition of locally conformal symplectic blow-up of points and proved that the locally conformal symplectic blow-ups of points also admit locally conformally symplectic structures. Therefore, a natural problem is: *What is the locally conformal symplectic blow-up along a submanifold?*

The purpose of this paper is to study some birational properties of locally conformal symplectic manifolds. Inspired by the work of McDuff [9] we give the construction of the locally conformal symplectic blow-up. In addition, using the same methods of McDuff [9] and Ornea–Verbitsky–Vuletescu [11] we prove the following result

Theorem 1.1. *Let (M, ω, θ) be a locally conformal symplectic manifold and Z be a compact induced globally conformal symplectic submanifold of M, and let $\pi : \tilde{M} \to M$ be the blow-up of M along Z. Then $\tilde{M}$ also admits a locally conformal symplectic structure $(\tilde{\omega}, \tilde{\theta})$ where $\tilde{\theta} = \pi^*\theta$.*

This paper is organized as follows. We devote Section 2 to the preliminary of locally conformal symplectic structures. In Section 3, we give the construction of locally conformal symplectic blow-up. This construction is based on the fact that the tangent bundle of a locally conformal symplectic manifold is a symplectic vector bundle. In Section 4, we give the proof of the main result (Theorem 1.1). Finally, we propose two further problems related to the locally conformal symplectic blow-up.