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On a classification of fat bundles over compact homogeneous spaces

Maciej Bocheński, Anna Szczepkowska, Aleksy Tralle *, Artur Woike

Department of Mathematics and Computer Science, University of Warmia and Mazury, Słoneczna 54, 10-710 Olsztyn, Poland

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ABSTRACT

This article deals with fat bundles. Bérard-Bergery classified all homogeneous bundles of that type. We ask a question of a possibility to generalize his description in the case of arbitrary G-structures over homogeneous spaces. We obtain necessary conditions for the existence of such bundles. These conditions yield a kind of classification of fat bundles associated with G-structures over compact homogeneous spaces provided that the connection in a G-structure is canonical.

1. Introduction

Let $P(M, G)$ be a principal bundle endowed with a connection form θ and its curvature form Ω. Let $\operatorname{Ker}\theta = \mathcal{H} \subset TP$ be the corresponding horizontal distribution. Assume that the Lie algebra $\mathfrak{g}$ of G is endowed with an invariant non-degenerate bilinear form $B_{\mathfrak{g}}$. We say that a vector $u \in \mathfrak{g}$ is *fat* or that the connection form θ is u-fat, if the bilinear two-form $B_{\mathfrak{g}}(\Omega(\cdot,\cdot), u)$ is non-degenerate on $\mathcal{H}$. If the fatness condition is fulfilled for every non-zero $u \in \mathfrak{g}$ then we say that the connection form θ is fat or that the principal bundle is fat. The role of the fatness condition in Riemannian geometry follows from its relation with the O'Neil tensor [2,18] of a specific fiberwise metric on the associated bundle. In greater detail, consider an associated bundle

$$F \to P \times_G F \to M.$$

* Corresponding author.
E-mail addresses: mabo@matman.uwm.edu.pl (M. Bocheński), anna.szczepkowska@matman.uwm.edu.pl (A. Szczepkowska), tralle@matman.uwm.edu.pl (A. Tralle), awoike@matman.uwm.edu.pl (A. Woike).

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Endow F with a G-invariant Riemannian metric g_F, and M with a Riemannian metric g_M. Equip $P \times_G F$ with the *connection metric* defining it to be the Riemannian metric which equals g_F on F, $g(X^*, Y^*) = g_M(X, Y)$ for horizontal lifts X^*, Y^* of $X, Y \in TM$ respectively, and declaring TF and $\mathcal{H}$ to be orthogonal with respect to g. For this metric the following holds. Let A_X denote the O'Neil tensor.

Theorem 1. [18] *The connection metric g on $P \times_G F$ is complete and defines a Riemannian submersion $\pi : P \times_G F \to M$ with totally geodesic fiber F and with holonomy group a subgroup of G. Conversely, every Riemannian submersion over M with totally geodesic fibers arises in this fashion. Moreover, the O'Neil tensor for such submersion satisfies the equality*

$$\theta(A_X Y) = -\Omega(X, Y).$$

The condition of fatness is an important tool of constructing manifolds of positive and non-negative curvature [9,17]. In the Riemannian context, the following definition of fatness is used: a Riemannian submersion $\pi : E \to M$ with totally geodesic fibers is fat, if $A_X U \neq 0$ for all horizontal vector fields X and vertical vector fields U ("all vertizontal curvatures are positive"). For the associated bundles, the characterization of fatness of any connection is known [18] and can be formulated as follows.

Theorem 2. [18] *The Riemannian submersion $P \times_G F \to M$ with totally geodesic fibers G/L is fat if and only if the 2-form $B_{\mathfrak{g}}(\Omega(X, Y), u)$ is non-degenerate on the horizontal distribution for all $u \in \mathfrak{l}^\perp$.*

Keeping the above theorem in mind, from now on whenever we say that an associated bundle $G/L \to P/L \to M$ is fat, we mean that the set $\mathfrak{l}^\perp$ consists of fat vectors. In this paper we consider associated bundles with homogeneous fibers since it was proved (see Proposition 2.6 in [18]) that every fat submersion necessarily has a homogeneous fiber. Note that the connection metric g depends on a (chosen) principal connection. The following is known.

1. There is an algebraic condition on the curvature tensor of the associated metric connection of the sphere bundles of the form

$$SO(n+1)/SO(n) \to P/SO(n) \to M$$

   ensuring fatness (see Proposition 2.21 in [18]).
2. A theorem in [5] shows that the only fat $SO(4)/SO(3)$-bundle over S^4 is the Hopf bundle $S^7 \to S^4$.
3. A theorem of Bérard-Bergery [3] which classifies all homogeneous fat bundles, that is, associated bundles of the form $H/L \to K/L \to K/H$, where K, H, L are compact Lie groups. Note that in this case the classification is obtained for *any* invariant connection.
4. There are necessary conditions for fatness, see for example [8].

To stress the fact that in general the fatness condition is dependent on the choice of the connection, we will always refer to "fatness with respect to a connection". Here it seems to be instructive to compare [3] with the general problem. In the case of principal bundles $K \to K/H$, there is a one-to-one correspondence between the invariant connections and the linear maps $\Lambda : \mathfrak{k} \to \mathfrak{h}$ satisfying the conditions $\Lambda(X) = X$ for any $X \in \mathfrak{h}$, and $\Lambda([X, Y]) = [X, \Lambda(Y)]$ for all $X \in \mathfrak{h}$, $Y \in \mathfrak{k}$. If, for example, $\mathfrak{k}$ is semisimple the fatness condition can be expressed as follows: for any $X \in \mathfrak{h}^\perp$ and any $Y \in \operatorname{Ker}\Lambda$ there exists $Z \in \operatorname{Ker}\Lambda$ such that $\langle X, \Lambda([Y, Z])\rangle \neq 0$. Here $\langle \cdot, \cdot \rangle$ denotes the Killing form. The latter condition can be expressed entirely in terms of the Lie brackets of $\mathfrak{k}$. In general, the problem becomes much more complicated even for the invariant connections in G-structures. In this work we will adopt the following terminology. A homogeneous bundle