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Deformation in holomorphic Poisson manifolds

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ABSTRACT

In this paper, we consider deforming a coisotropic submanifold Y in a holomorphic Poisson manifold (X, π). Under the assumption that Y has a holomorphic tubular neighborhood, we associate Y with an L_∞-algebra that controls the deformations of Y. This L_∞-algebra can also be extended to control the simultaneous deformations of the holomorphic Poisson structure π and the coisotropic submanifold Y.

1. Introduction

Given an algebraic or geometric structure Σ, one may consider a correction of Σ as a structure of the same kind, which we call a deformation of Σ. In certain situations, such deformations can be simply modeled as the Maurer–Cartan elements of a differential graded Lie algebra (DGLA), but in more general setups, DGLA is not sufficient. Gradually, L_∞-algebra becomes a fixed tool in providing algebraic description of deformations. An L_∞-algebra L_Σ controls the deformations of Σ if there is a 1-1 correspondence between the deformations of Σ and the Maurer–Cartan elements of L_Σ.

Recently, fruitful results have been obtained on deformation problems. Y. Oh and J. Park [9] construct an L_∞-algebra that controls the deformations of a coisotropic submanifold in a symplectic manifold. A.S. Cattaneo and G. Felder associate an L_∞-algebra (in fact P_∞-algebra) to any coisotropic submanifold S of a Poisson manifold (M, π) [2]. However, only under certain regularity conditions, this L_∞-algebra controls the coisotropic deformations of S [10]. In the case of deforming a Lie subalgebroid E of a Lie algebroid $(A, [\cdot,\cdot], \rho)$, the author constructs an L_∞-algebra associated with E, and proves that it controls the deformations of E under certain regularity conditions [7].

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In this paper, we consider the deformation problem of a coisotropic submanifold in a holomorphic Poisson manifold. We start with the more general setup – a coisotropic submanifold Y in an extended Poisson manifold X, and associate Y with an L_∞-algebra L_Y under the assumption that Y has a holomorphic tubular neighborhood. It turns out L_Y does not control the coisotropic deformations of Y in general. But in the particular case that X is holomorphic Poisson, an L_∞-subalgebra of L_Y can be constructed which controls the deformations of Y. Moreover, we can also control the simultaneous deformations of the holomorphic Poisson structure on X and the coisotropic submanifold Y.

Closely related with the question considered in this paper, the problem of formal deformations of a coisotropic submanifold in a holomorphic Poisson manifold is attacked in [1]. An L_∞-algebra that controls these formal deformations is constructed, but in a weaker sense that instead of requiring strictly isomorphic, they only require the corresponding Deligne-groupoid being equivalent to the groupoid of formal deformations. This L_∞-algebra does not require the existence of a holomorphic tubular neighborhood. However, it is quasi-isomorphic to the one we construct in Theorem 3.5 provided that a holomorphic tubular neighborhood exists and the extended Poisson structure degenerates to a holomorphic Poisson structure. Moreover, only formal deformations are consider in [1].

2. Preliminary

2.1. L_∞-algebras

We mainly follow [8], where readers may find more details. Let k be a field. A $\mathbb{Z}$-graded vector space over k is a direct sum $V = \oplus_{k\in\mathbb{Z}} V_k$ of k-vector spaces. An element v is homogeneous if $v \in V_k$ for some k, and its degree is $|v| = k$. A linear subspace of V is a subset $V' \subseteq V$, such that $V'_k = V' \cap V_k$ is subspace of V_k for all k. A linear map $f : V \to W$ between graded vector spaces is a collection of linear maps $\{f_k : V_k \to W_k\}_{k\in\mathbb{Z}}$. The direct sum and tensor product of two graded vector spaces V and W are again graded vector spaces:

$$(V \oplus W)_k = V_k \oplus W_k, \qquad (V \otimes W)_k = \oplus_{i+j=k} V_i \otimes W_j.$$

For any $n \in \mathbb{Z}$, the n-th suspension of V, denoted by $V[n]$, is a graded vector space with grading $(V[n])_k = V_{k+n}$. A linear map of degree n from V to W is a linear map $V \to W[n]$. For any $v \in V$, we use $v[n]$ to denote the corresponding element in $V[n]$.

Let S_k be the symmetric group of k letters. The symmetric action of S_k on $\otimes^k V$ is determined by

$$\sigma(v_1 \otimes \cdots \otimes v_k) = (-1)^{|v_j||v_{j+1}|} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

where $v_1, \cdots, v_k \in V$ are homogeneous, and $\sigma \in S_k$ transposes the j-th and $(j+1)$-th components. In general, the sign from the action of $\tau \in S_k$ on $v_1 \otimes \cdots \otimes v_k$ is denoted by $e(\tau)$. A linear map $l_k : \otimes^k V \to V$ is symmetric if

$$l_k(v_{\tau(1)} \otimes \cdots \otimes v_{\tau(k)}) = e(\tau) l_k(v_1 \otimes \cdots \otimes v_k).$$

Denote by $S_{j,n-j}$ the set of $(j, n-j)$ shuffles in S_n, i.e. $\tau \in S_{j,n-j}$ if and only if $\tau(1) < \cdots < \tau(j)$ and $\tau(j+1) < \cdots < \tau(n)$.

Definition 2.1. An L_∞-algebra is a $\mathbb{Z}$-graded vector space V together with a family of symmetric linear maps $\{m_k : \otimes^k V \to V[1]\}_{k\geq 0}$, such that the family of *Jacobiators* $\{J_n : \otimes^n V \to V[2]\}_{n\geq 1}$, defined by

$$J_n(v_1, \cdots, v_n) = \sum_{i+j=n+1} \sum_{\tau\in S_{j,n-j}} e(\tau) m_i(m_j(v_{\tau(1)}, \cdots, v_{\tau(j)}), v_{\tau(j+1)}, \cdots, v_{\tau(n)}),$$