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On Obata theorem in Randers spaces

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The classical Obata theorem in Riemannian space is generalized to the Randers spaces. It is proved that, if the generalized Obata equation on a closed Douglas Randers spaces admits a nontrivial solution, then M is weakly isometric to the Euclidean sphere $(\mathbb{S}^n(1), h)$, where, h denotes the standard Riemannian metric of $\mathbb{S}^n(1)$. In particular, F is locally projectively flat with positive flag curvature.

1. Introduction

The first eigenvalue λ_1 of the Laplacian operator on $(\mathbb{S}^n(\sqrt{k}), h)$ (the Euclidean sphere of radius $1/\sqrt{k}$ in $\mathbb{R}^{n+1}$) is nk and each eigenfunction f corresponding to nk satisfies the following system of differential equations (cf. [11]):

$$\nabla df + kfh = 0, \quad k > 0, \tag{1}$$

where, ∇ denotes the Levi-Civita connection of the induced Riemannian metric h from $\mathbb{R}^{n+1}$ on $\mathbb{S}^n(\sqrt{k})$. The equation (1) is relevant to the studies on the first eigenvalue of the Laplacian operator on Riemannian manifolds, since it provides that f is an eigenfunction for the first eigenvalue which is valid also in Finsler geometry, see the recent works [15,16]. Lichnérowicz proved in [9] that, given any n-dimensional closed Riemannian manifold with positive constant Ricci curvature k, we have $\lambda_1 \geq nk$ and Obata proved that, the sphere $\mathbb{S}^n(\sqrt{k})$ is unique among all of complete Riemannian manifolds for which the lower bound nk is achieved, cf. [11]. It should be noted that the lower bound nk is achieved on $\mathbb{S}^n(\sqrt{k})$ by restricting

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any linear function on $\mathbb{R}^{n+1}$ to the unit sphere $\mathbb{S}^n(\sqrt{k})$ (i.e. spherical harmonics of degree one). Any such eigenfunction f, is a solution of (1) and vice-versa, cf. [11]; Hence, solvability of (1) can be regarded as a characteristic equation for the Euclidean sphere $\mathbb{S}^n(\sqrt{k})$, (cf. [8]). Assuming a specific curvature condition, Akbar-Zadeh proved also the inequality $\lambda_1 \geq nk$ for the first eigenvalue of the horizontal Laplacian on the closed and simply connected Finsler manifolds, (cf. [1], p. 95), however, his result yields only homeomorphism to the Euclidean sphere $\mathbb{S}^n(1)$. Later, Gallot found in [7] several characteristic equations corresponding to each eigenvalue of the Laplacian on $(\mathbb{S}^n(\sqrt{k}), h)$. Given any nontrivial (i.e. nonconstant) solution f of (1), $\mathtt{grad}f$ is a conformal vector field. The characteristic equation (1) can be relaxed on a Riemannian manifold (M, h) to the form

$$\nabla df + \varphi f h = 0, \tag{2}$$

where, φ is an appropriate smooth real function. Up to conformal equivalence, Tashiro classified all connected and complete n-dimensional Riemannian manifolds on which the equation (2) has a nontrivial solution into three types: $\mathbb{S}^n(1)$, $\mathbb{R}^n$ and the warped product $J \times \bar{M}$ of an open interval and a complete Riemannian manifold $\bar{M}$, cf. [14]. Later on, Tanno completed the works of Obata and Tashiro, cf. [13]. A generalization of the equation (1) on Finsler spaces has been recently studied by Bidabad and Asanjarani in [3] and also in [4] to establish a Finslerian extension of Tashiro's classification of complete Riemannian spaces as well as Obata theorem. Here, we would like to study the equation (1) in a Randers space. We use the following better – in the sense that it depends only to the geodesic spray coefficient – and equivalent form of the system of PDEs given by (2) on a Finsler space (M, F), namely:

$$D_0 D_0 f + \varphi f F^2 = 0, \tag{3}$$

where, D denotes the Berwald connection and $D_0 = y^i D_i$ stands for the covariant derivative along the canonical geodesic spray. Notice that, the equation (3) is not sensitive to the change of affinely equivalent connections such as Chern, Berwald or Cartan connections; Moreover, equation (3) is equivalent to the following tensorial equation:

$$D_i D_j f + \varphi f g_{ij} = 0. \tag{4}$$

Given any Randers metric on M, denote the Levi-Civita connection of α by ∇ and recall the usual symbolic conventions for general (α, β)-metrics, cf. [6]. We give a characterization of the solutions of (3) by following result:

Theorem 1. *Let us suppose that $(M, F = \alpha + \beta)$ be a Randers space. The differential equation (3) has a nontrivial solution f if and only if the following three statements hold:*

(a) F is of isotropic S-curvature $S(x, y) = (n + 1)c(x)F(x, y)$,
(b) f is a solution of $c\nabla_0 f - \varphi f \beta + s^i{}_0 \nabla_i f = 0$,
(c) f is a solution of $\nabla_0\nabla_0 f + 2\nabla_0 f s_0 - 2\beta s^i{}_0 \nabla_i f - \varphi f(\alpha^2 + 3\beta^2) = 0$.

Two Finsler metrics F and $\tilde{F}$ on a manifold M are said to be *weakly isometric* if there is a vector field W on M such that the pair (F, W) solves the Zermelo navigation problem $\tilde{F}(x, \frac{y}{F} + W_x) = 1,\ x \in M,\ y \in T_x M$. Two Finsler manifold (M, F) and $(\tilde{M}, \tilde{F})$ are said to be weakly isometric if there is diffeomorphism $\phi : M \longrightarrow N$ such that F and $\phi^*\tilde{F}$ are weakly isometric. This terminology was first proposed by Zhongmin Shen and used in [12]. Theorem 1 implies the following result:

Theorem 2. *Let $(M, F = \alpha + \beta)$ be a closed and Douglas Randers spaces of dimension $n \geq 2$ expressed by the navigation data (h, W). If the PDE*