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Compact translating solitons with non-empty planar boundary

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ABSTRACT

In this paper, we consider compact translating solitons with non-empty planar boundary. Each boundary component lies in a plane which is orthogonal to the translating direction. We firstly prove that when the planar boundary is either a circle or convex and the translating soliton meets the plane containing the boundary with a constant angle, then the compact translating soliton is part of an entire rotationally symmetric strictly convex graphical surface. Secondly, we show that a compact translating soliton spanning two horizontal planar Jordan curves inherits the symmetries of its boundary. We also show a balancing type formula for compact translating solitons with planar boundary.

1. Introduction

For a compact surface with boundary, the study of influence of the boundary on the surface is a classical topic in differential geometry. The simplest compact boundary is clearly a circle. One of long-lasting interesting problems about circular boundary is *the spherical cap conjecture*:

A constant mean curvature surface $\Sigma \subset \mathbb{R}^3$ with a circular boundary is a planar disc or a spherical cap when Σ is either an immersed disc or a compact embedded surface.

There are some interesting partial results on this conjecture [5,10,12,13]. In [4], Hoffman, Rosenberg and Spruck showed that a compact constant Gaussian curvature surface spanning a circle is a spherical cap.

In this paper, we study the rigidity of translating solitons spanning a planar boundary (including a circular boundary).

Let $x : \Sigma^n \to \mathbb{R}^{n+1}$ be an n-dimensional isometric immersion, for simplicity we identify $x(\Sigma)$ with Σ. For a unit vector $v \in \mathbb{R}^{n+1}$, Σ is a *translating soliton* of the unit speed with respect to the translating direction v if $X(p,t) = x(p) + tv : \Sigma \times \mathbb{R} \to \mathbb{R}^{n+1}$ satisfies the following mean curvature flow equation

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$$X_t(p,t) = \vec{H}(X(p,t)),$$

for all $(p,t) \in \Sigma \times \mathbb{R}$, that is, Σ satisfies $H = \langle N, v\rangle$, where $\vec{H} = HN$ is the mean curvature vector.

Translating solitons arise as parabolic rescaling of Type-II singularities of the mean curvature flow [8].

The simplest nonplanar translating soliton is an entire rotationally symmetric strictly convex graphical hypersurface, U, found by Altschler and Wu [1] for $n = 2$, and Gui, Jian and Ju [6] for $n \geq 3$. It is unique up to translation.

From now on, by using an isometry, we assume that $v = e_{n+1} = (0, \dots, 1)$. Denoting the distance function, $r = \sqrt{x_1^2 + \cdots + x_n^2}$ for $n \geq 2$, U is the graph of the following ordinary differential equation with respect to r, $x_{n+1} = u(r)$,

$$\frac{u''(r)}{1 + u'(r)^2} + (n-1)\frac{u'(r)}{r} = 1,$$

with $u(0) = u'(0) = 0$. Moreover, as $r \to \infty$

$$u(r) = \frac{r^2}{2(n-1)} - \ln r + O(r^{-1}),$$

that is, $U = \mathit{Graph}(u)$ is asymptotic to a parabola.

We note that Clutterbuck, Schnürer and Schulze [3] showed that there exist non-convex rotationally symmetric translating solitons in $\mathbb{R}^3$ named *winglike* translating solitons. Nguyen [14,15] constructed various complete embedded translating solitons in $\mathbb{R}^3$ using a gluing technique.

Now we consider compact translating solitons with non-empty planar boundary. From the view point of rotationally symmetric surfaces, we show that the compact part of U separated by a horizontal plane is characterized as follows:

Main Theorem. *Let U be the entire rotationally symmetric strictly convex graphical translating soliton in $\mathbb{R}^3$. Let Σ be a compact connected immersed translating soliton with smooth boundary $\Gamma = \partial\Sigma$ lying in a horizontal plane $\Pi = \{(x,y,z) | z = c \in \mathbb{R}\}$.*

(1) *If Γ is a circle, then Σ is part of U.*
(2) *If Γ is convex and Σ meets Π with a constant angle along Γ, then Σ is also part of U.*

When a compact translating soliton Σ is graphical over a domain D in a horizontal plane, then there exists a function $u(x,y)$ so that $\mathit{Graph}(u) = \Sigma$ and it satisfies

$$\sqrt{1 + |\nabla u|^2}\, div \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 1,$$

for all $(x,y) \in D$. Serrin [17, §20] showed that the necessary and sufficient conditions for existence of such a function are that the domain D is mean convex. So there are plenty of compact translating solitons with planar boundary which are different from a compact part of U separated by a horizontal plane. We note that Serrin's result holds for higher dimension.

2. Proof of the Main Theorem

A fundamental tool to prove the Main Theorem is the tangency principle based on maximum principle of elliptic partial differential equations.