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Principal bundle structures among second order frame bundles

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Using a model for the bundle $\hat{\mathcal{F}}^2 M$ of semi-holonomic second order frames of a manifold M as an extension of the bundle $\mathcal{F}^2 M$ of holonomic second order frames of M, we introduce in $\hat{\mathcal{F}}^2 M$ a principal bundle structure over $\mathcal{F}^2 M$, the structure group being the additive group $A_2(n)$ of skew-symmetric bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$. The composition of the projection of that structure with the existing projection of the bundle $\tilde{\mathcal{F}}^2 M$ of non-holonomic second order frames of M over $\hat{\mathcal{F}}^2 M$ provides a principal bundle structure in $\tilde{\mathcal{F}}^2 M$ over $\mathcal{F}^2 M$. These results close an existing gap in the theory of second order frame bundles.

1. Introduction

In the recent years, higher order frames have become an important tool that has contributed to the development of different chapters both inside the framework of differential geometry and physics. We can cite for example differential invariants, classical field theories, quantum field theory, continuum mechanics [1,4,12]. Since they were introduced by Ehresmann in 1995, non-holonomic and semi-holonomic higher order frame bundles have known several equivalent interpretations, most of them in terms of jet bundles. The characterization given here as a starting point is the one followed in the work by Elzanowski and Prishepionok [3], although later we make use of the interpretation of the non-holonomic and semi-holonomic second order frame bundles $\tilde{\mathcal{F}}^2 M$ and $\hat{\mathcal{F}}^2 M$ of a differentiable manifold M as extensions of the holonomic second order frame bundle $\mathcal{F}^2 M$ [11].

One kind of problems that arises in the context of these theories is the structure problems. Recently, Brajerčík, Demko and Krupka [2] have realized a construction of a principal bundle structure on the r-jet

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prolongation of the linear frame bundle of an n-dimensional manifold; such prolongations involve semi-holonomic second order frames [8]. In this paper we also deal with a structure problem, raised in [10], for second order frames: the possible existence of a principal bundle structure on $\tilde{\mathcal{F}}^2M$ and $\hat{\mathcal{F}}^2M$ over $\mathcal{F}^2M$. The answer to the question is in the affirmative and the way in which we have treated it is a constructive way, in the sense that we have defined in detail such structures. The main drawback lies in finding a suitable projection $\hat{\pi}^2_2\colon \hat{\mathcal{F}}^2M \to \mathcal{F}^2M$. An early work by Kolář [6] provides a symmetrization of semi-holonomic 2-jets using exact diagrams of vector bundles and splitting properties. Some other attempt of symmetrization in higher order using additional geometric background has only reached a relative success [13]. The idea underlying in our definition of the projection $\hat{\pi}^2_2$ lies in the fact that the description of the bundle $\hat{\mathcal{F}}^2M$ as an extension of the bundle $\mathcal{F}^2M$ allows an algebraic manner to symmetrize the bilinear part of second order jets in a global way, on the basis that the symmetrization of a (semi-holonomic) jet somehow commutes with the composition with holonomic jets. The provided solution also completes the problem of the existence of principal bundle structures among the three second order frame bundles $\tilde{\mathcal{F}}^2M$, $\hat{\mathcal{F}}^2M$ and $\mathcal{F}^2M$, the linear frame bundle $\mathcal{F}M$, and the own manifold M. These relations are summarized in the Section 5 in the final part of the paper.

2. Preliminaries

We are going to deal with five differentiable manifolds: an n-dimensional manifold M, its linear frame bundle $\mathcal{F}M$ and the bundles $\mathcal{F}^2M$, $\hat{\mathcal{F}}^2M$ and $\tilde{\mathcal{F}}^2M$ of holonomic, semi-holonomic and non-holonomic second order frames of M, respectively. Let us denote by π^1_0, π^2_0, $\hat{\pi}^2_0$ and $\tilde{\pi}^2_0$ the natural projections of $\mathcal{F}M$, $\mathcal{F}^2M$, $\hat{\mathcal{F}}^2M$ and $\tilde{\mathcal{F}}^2M$ on M and by $G^1(n) = GL(n,\mathbb{R})$, $G^2(n)$, $\hat{G}^2(n)$ and $\tilde{G}^2(n)$ their respective structure groups. The second order frames can be described as follows. Let $\tilde{\phi}\colon \mathbb{R}^n \to \mathcal{F}M$ be a differentiable map such that $\pi^1_0 \circ \tilde{\phi}$ is a diffeomorphism. The 1-jet $j^1_0\tilde{\phi}$ with source at the origin $0 \in \mathbb{R}^n$ and target in $\mathcal{F}M$ is a non-holonomic second order frame at the point $x = \pi^1_0(\tilde{\phi}(0)) \in M$. If $\tilde{\phi}$ satisfies the additional condition $\tilde{\phi}(0) = j^1_0(\pi^1_0 \circ \tilde{\phi})$, then $j^1_0\tilde{\phi}$ is called semi-holonomic. A holonomic second order frame is an invertible 2-jet $j^2_0 f$ where f is a differentiable map from $\mathbb{R}^n$ into M with source at $0 \in \mathbb{R}^n$ and target in M; a holonomic second order frame can be also seen as a semi-holonomic second order frame $j^1_0\tilde{\phi}$ verifying $\tilde{\phi} = (\pi^1_0 \circ \tilde{\phi})^{(1)} \circ \eta_1$, where $\eta_1\colon \mathbb{R}^n \to \mathcal{F}\mathbb{R}^n \equiv \mathbb{R}^n \oplus G^1(n)$ is the section given by $\eta_1 = \mathrm{id}_{\mathbb{R}^n} \times I$ and, if M_1 and M_2 are differentiable manifolds, $F^{(1)}\colon j^1_0 g \in \mathcal{F}M_1 \to j^1_0(F \circ g) \in \mathcal{F}M_2$ is the prolonged map of a local diffeomorphism $F\colon M_1 \to M_2$ between the corresponding linear frame bundles. In order to describe second order frames in local coordinates [7,11], we put $\tilde{\phi}(r^a) \equiv (\phi^i(r^a), \phi^i_j(r^a))$, where $\{r^a\}$ are the natural coordinates in $\mathbb{R}^n$. Then the non-holonomic frame $j^1_0\tilde{\phi}$ is expressed as

$$\left(\phi^i(0), \phi^i_j(0), \frac{\partial \phi^i}{\partial r^j}(0), \frac{\partial \phi^k_l}{\partial r^j}(0)\right) \equiv (x^i, x^i_j, y^i_j, x^{kl}_j).$$

Here, (x^i, x^i_j) are the coordinates of $\tilde{\phi}(0)$ and (y^i_j, x^{kl}_j) are given by

$$\tilde{\phi}_*(0)\frac{\partial}{\partial r^j}(0) = y^i_j \frac{\partial}{\partial x^i}(z) + x^{kl}_j \frac{\partial}{\partial x^k_l}(z), \quad z = \tilde{\phi}(0), \quad j = 1, \ldots, n,$$

with (y^i_j) non-singular since $\pi^1_0 \circ \tilde{\phi}$ is a diffeomorphism. The semi-holonomic condition above results in $\phi^i_j(0) = \frac{\partial \phi^i}{\partial r^j}(0)$ (i.e., $x^i_j = y^i_j$) what translates into a system of local coordinates for $\hat{\mathcal{F}}^2M$ of the form (x^i, x^i_j, x^{kl}_j). On the other hand, the holonomic condition leads to $\frac{\partial \phi^k_l}{\partial r^j}(0) = \frac{\partial^2 \phi^k}{\partial r^l \partial r^j}(0)$, i.e., $x^{kl}_j = x^{kj}_l$.

Information on another models for non-holonomic and semi-holonomic second order frames can be found, for example, in [4,9–11,14].

Let $L_2(n)$ be the additive group of bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$ and let $S_2(n)$ and $A_2(n)$ be the additive subgroups of symmetric bilinear maps and skew-symmetric bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$