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Homogeneous spaces, curvature and cohomology

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ABSTRACT

We give new counterexamples to a question of Karsten Grove, whether there are only finitely many rational homotopy types among simply connected manifolds satisfying the assumptions of Gromov's Betti number theorem. Our counterexamples are homogeneous Riemannian manifolds, in contrast to previous ones. They consist of two families in dimensions 13 and 22. Both families are nonnegatively curved with an additional upper curvature bound and differ already by the ring structure of their cohomology rings with complex coefficients. The 22-dimensional examples also admit almost nonnegative curvature operator with respect to homogeneous metrics.

1. Introduction

In this paper we give new counterexamples to a question raised by Grove [11] whether there are only finitely many rational homotopy types of closed, simply connected manifolds with a given lower bound for the sectional curvature and a given upper bound on the diameter. Note that if in this question one replaces rational homotopy type by (integral) homotopy type, then the Aloff–Wallach spaces [1] give rise to counterexamples.

Counterexamples to Grove's questions have been constructed by Fang and Rong [8] and Totaro [15]. All these examples actually already differ by the ring structure of their rational cohomology rings and, except for Totaro's six dimensional family, each family has a uniform upper bound on the sectional curvature. Totaro's examples in dimensions 6 and 9 have the merit of being nonnegatively curved, while the examples of Fang and Rong differ by their cohomology rings with complex coefficients.

We show that there are counterexamples to Grove's question with an additional upper curvature bound among nonnegatively curved homogeneous spaces in dimensions 13 and ≥ 15, which differ by their complex

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cohomology rings. We also show that in dimensions 22 and ≥ 24 there are examples which in addition have almost nonnegative curvature operator with respect to homogeneous metrics.

Note that of Totaro's examples only finitely many can be homogeneous, as one can see from the classification of closed, simply connected, homogeneous spaces up to dimension 9 carried out by Klaus in his diploma thesis [12]. In the nine-dimensional case, from the classification of Klaus one sees that, in order to be homogeneous, Totaro's nine-dimensional examples would have to be principal circle bundles over the product $(\mathrm{S}^2)^4 = \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2$. These fall into finitely many rational homotopy types.

Theorem 1. *For $n = 13$ and $n \geq 15$ there exist $C, D > 0$ and an infinite family of closed, simply connected, homogeneous n-manifolds M_a with normal homogeneous metrics, such that their cohomology rings $\mathrm{H}^*(M_a; \mathbb{C})$ with complex coefficients are pairwise non-isomorphic, their sectional curvature satisfies $0 \leq \sec_{M_a} \leq C$ and their diameter $\operatorname{diam}(M_a) \leq D$.*

Bounds on the curvature operator of a manifold are much stronger than bounds on the sectional curvature. For instance, the only closed, simply connected manifolds with positive curvature operator are spheres by Böhm and Wilking [3]. This also finished the classification of simply connected manifolds with nonnegative curvature, see e.g. [6]. In particular, in a fixed dimension there are only finitely many diffeomorphism types of closed, simply connected manifolds with nonnegative curvature operator and Grove's question has a positive answer, if one replaces the lower sectional curvature bound by non-negativity of the curvature operator.

Recall that a closed manifold M has *almost nonnegative curvature operator* if for every $\varepsilon > 0$ there is a Riemannian metric g_ε on M whose curvature operator $\hat{R}_{g_\varepsilon}$ satisfies

$$\hat{R}_{g_\varepsilon}(\operatorname{diam}(M, g_\varepsilon))^2 > -\varepsilon.$$

We say that for a family $(g_t)_{t\in(0,1]}$ on M the family (M, g_t) has almost nonnegative curvature operator as $t \to 0$, if $\lim_{t\to 0} \lambda_{\min}(\hat{R}_{g_t})(\operatorname{diam}(M, g_t))^2 \geq 0$, where $\lambda_{\min}(\hat{R}_{g_t})$ is the smallest eigenvalue of $\hat{R}_{g_t}$.

In [10] first examples of closed, simply connected manifolds with almost nonnegative curvature that do not admit a metric of nonnegative curvature operator were constructed. Furthermore, also in [10] it was shown that there is an infinite family of closed, simply connected 6-manifolds which admit almost nonnegative curvature operator and have distinct (integral) homotopy types.

By looking at principal torus bundles over $(\mathbb{CP}^2)^5$ and their products with spheres we prove the following theorem.

Theorem 2. *For $n = 22$ and $n \geq 24$ there exist $C, D > 0$ and infinitely many homogeneous, closed, simply connected n-manifolds E_α with pairwise non-isomorphic complex cohomology rings, such that on every E_α there exists a family g_t, $t \in (0, 1]$, of homogeneous Riemannian metrics with*

- $0 \leq \sec_{g_1} \leq C$, $\operatorname{Ric}_{g_1} > 0$ *and* $\operatorname{diam}_{g_1} \leq D$*;*
- $0 \leq \sec_{g_t}$, $\operatorname{Ric}_{g_t} > 0$ *and* $\operatorname{diam}_{g_t} \leq D$ *for every* $t \in (0, 1]$ *and* (E_α, g_t) *has almost nonnegative curvature operator as* $t \to 0$*.*

The metric g_1 is in fact normal homogeneous with respect to the transitive $\mathrm{SU}(3)^5 \times \mathrm{T}^2$ action on the E_α. To our knowledge, this family gives the first examples of closed, simply connected manifolds in a fixed dimension with almost nonnegative curvature operator and infinitely many different rational homotopy types.

Note that in both theorems, the higher dimensional families are obtained from the families in dimensions 13 and 22 by taking products with spheres. By taking products with a circle, one can also obtain families in dimensions 14 and 23, respectively, that share the same properties, except for being simply connected.