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Weak measure expansive flows

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Abstract

A notion of measure expansivity for flows was introduced by Carrasco-Olivera and Morales in [3] as a generalization of expansivity, and they proved that there were no measure expansive flows on closed surfaces. In this paper we introduce a concept of weak measure expansivity for flows which is really weaker than that of measure expansivity, and show that there is a weak measure expansive flow on a closed surface. Moreover we show that any C^1 stably weak measure expansive flow on a C^∞ closed manifold M is Ω-stable, and any C^1 stably measure expansive flow on M satisfies both Axiom A and the quasi-transversality condition.

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Keywords: Expansive; Flows; Measure expansive; Weak measure expansive; C^1-stably weak measure expansive

1. Introduction

The notion of expansivity for a homeomorphism on a compact metric space introduced by Utz [15] is important in the qualitative study of dynamical systems. Roughly speaking, a system is expansive if two orbits cannot remain close to each other under the action of the system. In light of the rich consequence of expansiveness in the dynamics of a system, it is natural to consider another notion of expansiveness. Recently Morales [6] introduced the notion of *measure expansiveness*, generalizing the usual concept of expansiveness. Several interesting properties of

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measure expansiveness have been obtained elsewhere [1,10,13,14]. In particular, Artigue and Carrasco-Olivera [1] characterized the homeomorphisms for which all probability measures are expansive as those which are *countably-expansive*.

On the other hand, Bowen and Walters [2] proposed an extension of expansivity to flow and proved that certain properties in the discrete case hold true in the flow's context too. Ruggiero [12] considered slightly more general definitions whereas Komuro [4] introduced a kind of expansivity allowing non-isolated singularities. More general extension called measure expansivity using Borel measures on a compact metric space was introduced by Carrasco-Olivera and Morales in [3], and they proved that there were no measure expansive flows on closed surfaces.

In this paper we introduce a notion of weak measure expansivity for flows which is really weaker than that of measure expansivity, and show that there is a weak measure expansive flow on a closed surface. Furthermore we study the relationship between stably measure expansivity and hyperbolicity for flows. More precisely, we show that any C^1 stably weak measure expansive flows on a C^∞ closed manifold M is Ω-stable, and any C^1 stably measure expansive flow on M satisfies both Axiom A and the quasi-transversality condition. Note that the proof in [1] only works for homeomorphisms, but not for flows.

Let (X, d) be a compact metric space. A *flow* on X is a continuous map $\phi : X \times \mathbb{R} \longrightarrow X$ satisfying $\phi(x, 0) = x$ and $\phi(\phi(x, s), t) = \phi(x, s + t)$ for $x \in X$ and $s, t \in \mathbb{R}$. For convenience, we will denote

$$\phi(x, s) = \phi_s(x) \quad \text{and} \quad \phi_{(a,b)}(x) = \{\phi_t(x) : t \in (a, b)\}.$$

The set $\phi_{\mathbb{R}}(x)$ is called the *orbit of* ϕ *through* $x \in X$ and will be denoted by $\mathcal{O}_\phi(x)$.

We say that a flow ϕ on X is *expansive* if for any $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in X$ satisfy $d(\phi_t(x), \phi_{h(t)}(y)) \le \delta$ for some $h \in \mathcal{H}$ and all $t \in \mathbb{R}$ then $y \in \phi_{(-\epsilon,\epsilon)}(x)$, where $\mathcal{H}$ denotes the set of continuous maps $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$. For any flow ϕ on X, $x \in X$ and $\delta > 0$, we denote $\Gamma^\phi_\delta(x)$ by

$$\{y \in X : d(\phi_t(x), \phi_{h(t)}(y)) \le \delta \text{ for some } h \in \mathcal{H} \text{ and all } t \in \mathbb{R}\},$$

and it is called *the dynamical* δ*-ball* of ϕ centered at $x \in X$. Note that $\Gamma^\phi_\delta(x)$ can be expressed by

$$\Gamma^\phi_\delta(x) = \bigcup_{h\in\mathcal{H}} \bigcap_{t\in\mathbb{R}} \phi_{-h(t)}(B[\phi_t(x), \delta]),$$

where $B[x, \delta]$ denotes the closed δ-ball centered at x.

Let $\mathcal{M}(X)$ be the set of all Borel probability measures μ on X, and $\mathcal{M}^*(X)$ the set of nonatomic measures μ in $\mathcal{M}(X)$. Denote by $\mathcal{M}^*_\phi(X)$ the set of μ in $\mathcal{M}(X)$ vanishing along the orbits of the flow ϕ on X. More precisely, we let

$$\mathcal{M}^*_\phi(X) = \{\mu \in \mathcal{M}(X) : \mu(\mathcal{O}_\phi(x)) = 0 \text{ for all } x \in X\}.$$

Then we have $\mathcal{M}^*_\phi(X) \subset \mathcal{M}^*(X) \subset \mathcal{M}(X)$.

For any subset $B \subset X$ (Borel measurable or not) we write $\mu(B) = 0$ if $\mu(A) = 0$ for any Borel subset $A \subset B$. For any $\mu \in \mathcal{M}(X)$, we say that ϕ is μ*-expansive* (or μ is *expansive* for ϕ) if there exists a constant $\delta > 0$ such that $\mu(\Gamma^\phi_\delta(x)) = 0$ for all $x \in X$. In the case, we sometimes say that