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Existence and relaxation of solutions for a subdifferential inclusion with unbounded perturbation

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ABSTRACT

An evolution inclusion is considered in a separable Hilbert space. The right-hand side of the inclusion contains the subdifferential of a time-dependent proper convex lower semicontinuous function and a multivalued perturbation with nonempty closed, not necessarily, bounded values. Along with this inclusion we consider the inclusion with the perturbation term being convexified. We prove the existence of solutions and a density of the solution set of the original inclusion in a closure of the solution set of the inclusion with the convexified perturbation. In contrast to the known results of this kind we do not suppose that the convex function has the compactness property and that the values of the perturbation are bounded sets. An example of a perturbation with closed unbounded values satisfying the conditions of the main theorems is given.

1. Introduction

Let T be the interval $[0,1]$ of the real half-line $\mathbb{R}^+ = [0,+\infty)$, $\overline{\mathbb{R}} = (-\infty,+\infty]$, and H be a separable Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. A function $\varphi : H \to \overline{\mathbb{R}}$ is called proper if its effective domain $\operatorname{dom}\varphi = \{x \in H; \varphi(x) < +\infty\}$ is nonempty. The class of all proper convex and lower semicontinuous functions $\varphi : H \to \overline{\mathbb{R}}$ is denoted by $\Gamma_0(H)$. For given function $\varphi \in \Gamma_0(H)$ and point x we denote by $\partial\varphi(x)$ the subdifferential of φ at x:

$$\partial\varphi(x) = \{v \in H; \langle v, y - x\rangle \le \varphi(y) - \varphi(x), \quad \forall y \in H\}. \tag{1.1}$$

It is known [2] that $\partial\varphi$ is a maximal monotone operator,

$$\operatorname{dom}\partial\varphi = \{x \in H; \partial\varphi(x) \ne \emptyset\} \subset \operatorname{dom}\varphi$$

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and

$$\overline{\operatorname{dom}\partial\varphi} = \overline{\operatorname{dom}\varphi},$$

where the bar stands for the closure in H. Let $cl(H)$ be the family of all nonempty closed sets from H and $U : T \times H \to cl(H)$ be a multivalued mapping. We consider the evolution inclusions

$$-\dot{x} \in \partial\varphi^t(x) + U(t,x), \tag{1.2}$$

$$-\dot{x} \in \partial\varphi^t(x) + \overline{\operatorname{co}}\, U(t,x), \tag{1.3}$$

$$x(0) = x_0 \in \operatorname{dom}\varphi^0,$$

where $\varphi^t \in \Gamma_0(H)$, $t \in T$ and the symbol $\overline{\operatorname{co}}$ denotes the closed convex hull of a set. The multivalued mapping $U(t,x)$ is usually called a perturbation. When the values of the mapping $U(t,x)$ are closed bounded (unbounded) sets the inclusion (1.2) is called an evolution inclusion with bounded (unbounded) perturbation. The evolution inclusion (1.3) is called an evolution inclusion with convexified perturbation.

By a solution of the inclusion (1.2) we mean a pair $(x(u), u)$ such that $x(u) \in W^{1,2}(T,H)$, $x(u)(0) = x_0$, $x(u)(t) \in \operatorname{dom}\partial\varphi^t$ a.e., $u \in L^2(T,H)$ and

$$-\dot{x}(u)(t) \in \partial\varphi^t(x(u)(t)) + u(t) \quad \text{a.e.}, \tag{1.4}$$

$$u(t) \in U(t, x(u)(t)) \quad \text{a.e.} \tag{1.5}$$

A solution of the inclusion (1.3) is defined similarly replacing $U(t,x)$ with $\overline{\operatorname{co}}\, U(t,x)$. In the present paper we study the existence of solutions of the unbounded evolution inclusion (1.2) and an approximation of solutions of the inclusion (1.3) by solutions of (1.2). This property is usually called the relaxation. Taking account of (1.4), (1.5) the evolution inclusion (1.2) can be considered as a simple control system with the control constraint $U(t,x)$.

Existence and relaxation of solutions of control systems with subdifferential operators and with a bounded mapping $U(t,x)$ were studied in the articles [13,14,16,18, and others]. When proving relaxation theorems in these works and other works of a similar kind, it was assumed that the function φ^t, $t \in T$ has the compactness property, i.e. for any $r \geq 0$, $t \in T$ the set

$$\{x \in H;\ \|x\| \leq r,\ |\varphi^t(x)| \leq r\}$$

is relatively compact in H. Existence of solutions of evolution inclusions and control systems with an unbounded mapping $U(t,x)$ in an infinite dimensional space was investigated in the articles [9,19,12,15]. Existence and relaxation of solutions of differential equations with unbounded right-hand sides in a finite dimensional space were proven in the works [6,20]. However, in an infinite dimensional space the questions of relaxation of solutions for both differential inclusions with unbounded right-hand side, evolution inclusions with unbounded perturbation and control systems with unbounded control constraints still remain open. The interest for such questions stems from the fact that unboundedness of values of multivalued mappings is a fairly natural property of inclusions encountered in the optimal control theory [8]. In the works [9,19,12,15], to prove the theorems on existence of solutions the author used a fixed point theorem for multivalued mappings with closed nonconvex decomposable values in the space of integrable functions [10]. In the present work we apply the scheme originated in [3] to prove existence and relaxation theorems for differential inclusions in a finite dimensional space, which we modify to suit an infinite dimensional space and the class of evolution inclusions which we consider. This allows us to prove in a uniform manner both existence and relaxation theorems without the compactness assumption for the functions φ^t, $t \in T$ and to obtain a priori estimates for solutions similar to those in