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Oscillation and nonoscillation results for solutions of half-linear equations with deviated argument

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ABSTRACT

We present oscillatory and nonoscillatory criteria for solutions of half-linear equations with deviated argument. Our method relies on the weighted Hardy inequality and differs from ones presented in the literature so far.

1. Introduction

Let us consider the *half-linear equation* with *deviated argument*

$$\left(r(t)|u'(t)|^{p-2}u'(t)\right)' + c(t)|u(\tau(t))|^{p-2}u(\tau(t)) = 0, \qquad t \in (0, \infty), \tag{1.1}$$

where $p > 1$, $c : [0, \infty) \to (0, \infty)$ is continuous, $c \in L^1(0,\infty)$, $r : [0, \infty) \to (0, \infty)$ is continuously differentiable, $\tau : [0, \infty) \to \mathbb{R}$ is continuously differentiable and increasing function satisfying $\lim\limits_{t\to\infty} \tau(t) = \infty$.

Assume that (1.1) has at least one nonzero global solution (see Section 2 for definition). We say that a global solution of (1.1) is *nonoscillatory* (at ∞) if there exists $T > 0$ such that $u(t) \neq 0$ for all $t > T$. Otherwise, it is called *oscillatory*, i.e., there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} t_n = \infty$ and $u(t_n) = 0$ for all $n \in \mathbb{N}$.

We prove the following two results. Note that $p' := \frac{p}{p-1}$.

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Theorem 1.1 *(nonoscillatory criterion). Let*

$$\limsup_{t\to\infty}\left(\int_0^t r^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_t^\infty c(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} < \frac{p-1}{p^{p'}} \tag{1.2}$$

and

$$\limsup_{t\to\infty}\left(\int_0^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_t^\infty c(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} < \frac{p-1}{p^{p'}}. \tag{1.3}$$

Then every global solution of (1.1) is nonoscillatory.

Theorem 1.2 *(oscillatory criterion). Let one of the following three cases occur:*

(i) There exists $T>0$ such that for all $t \ge T$ we have $\tau(t)\ge t$ and

$$\limsup_{t\to\infty}\left[\left(\int_0^t r^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_t^\infty c(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} + \left(\int_t^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_{\tau(t)}^\infty c(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}}\right] > 1.$$

(ii) There exists $T>0$ such that for all $t \ge T$ we have $\tau(t)\le t$ and

$$\limsup_{t\to\infty}\left(\int_0^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_t^\infty c(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} > 1.$$

(iii) For any $T>0$ the function $\tau(t)-t$ changes sign in (T,∞) and either

$$\liminf_{\substack{t\to\infty\\ t>\tau(t)}}\left(\int_0^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_t^\infty c(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} > 1$$

or

$$\liminf_{\substack{t\to\infty\\ t<\tau(t)}}\left[\left(\int_0^t r^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_t^\infty c(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} + \left(\int_t^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_{\tau(t)}^\infty c(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}}\right] > 1.$$

Then every global solution of (1.1) is oscillatory.

A typical example of $\tau=\tau(t)$ is a linear function

$$\tau(t)=t-\tau, \quad \tau\ge 0 \quad \text{is fixed.}$$

Then (1.1) is half-linear equation with the *delay* given by fixed parameter $\tau \ge 0$. For this rather special case, (1.2) implies (1.3) and only the case (ii) of Theorem 1.2 occurs. Hence we have the following corollary concerning the equation