J. Math. Anal. Appl. 447 (2017) 550–563

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Fully measurable small Lebesgue spaces

Giuseppina Anatriello [a], Maria Rosaria Formica [b], Raffaella Giova [b,*]

[a] *Dipartimento di Architettura, Università degli Studi di Napoli "Federico II", via Monteoliveto 3, 80134, Napoli, Italy*
[b] *Università degli Studi di Napoli "Parthenope", via Generale Parisi 13, Palazzo Pacanowsky, 80132, Napoli, Italy*

ARTICLE INFO

Article history:
Received 11 May 2016
Available online 21 October 2016
Submitted by B. Bongiorno

Keywords:
Banach function spaces
Grand and small Lebesgue spaces
Measurable exponent
Hölder-type inequality

ABSTRACT

We build a new class of Banach function spaces, whose function norm is

$$\rho_{(p[\cdot],\delta[\cdot]}(f) = \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in(0,1)} \rho_{p(x)}(\delta(x)^{-1} f_k(\cdot)),$$

where $\rho_{p(x)}$ denotes the norm of the Lebesgue space of exponent $p(x)$ (assumed measurable and possibly infinite), constant with respect to the variable of f, and δ is measurable, too. Such class contains some known Banach spaces of functions, among which are the classical and the small Lebesgue spaces, and the Orlicz space $L(\log L)^{\alpha}$, $\alpha > 0$.
Furthermore we prove the following Hölder-type inequality

$$\int_0^1 fg dt \leq \rho_{p[\cdot]),\delta[\cdot]}(f)\ \rho_{(p'[\cdot],\delta[\cdot]}(g),$$

where $\rho_{p[\cdot]),\delta[\cdot]}(f)$ is the norm of fully measurable grand Lebesgue spaces introduced by Anatriello and Fiorenza in [2]. For suitable choices of $p(x)$ and $\delta(x)$ it reduces to the classical Hölder's inequality for the spaces $EXP_{1/\alpha}$ and $L(\log L)^{\alpha}$, $\alpha > 0$.

1. Introduction

In [26] Iwaniec and Sbordone introduced the grand Lebesgue spaces $L^{p)}(\Omega)$ $(1 < p < \infty)$, $\Omega \subset \mathbb{R}^n$ of finite measure, in connection with the study of the integrability properties of the Jacobian determinant. In

* Corresponding author.
E-mail addresses: anatriello@unina.it (G. Anatriello), mara.formica@uniparthenope.it (M.R. Formica), raffaella.giova@uniparthenope.it (R. Giova).

http://dx.doi.org/10.1016/j.jmaa.2016.10.034

the case $\Omega = I = (0,1)$ such spaces are defined as the Banach function spaces (see e.g. [4] for the definition) of the measurable functions f on I such that

$$\|f\|_{p)} = \sup_{0<\epsilon<p-1} \left(\epsilon \int_0^1 |f(t)|^{p-\epsilon}\, dt \right)^{\frac{1}{p-\epsilon}} < \infty.$$

Since then the grand Lebesgue spaces play an important role in PDE's theory (see e.g. [18,23,25]), in Function Spaces theory (see e.g. [1,12,28,30,32,33]) and in interpolation–extrapolation theory (see e.g. [8, 15,24]). They have been widely investigated and several variations have been studied, among which, in [7], the spaces

$$L^{p),\delta}(I) = \left\{ f : I \to \mathbb{R} \text{ measurable } : \|f\|_{p),\delta} = \sup_{0<\epsilon<p-1} \left(\delta(\epsilon) \int_0^1 |f(t)|^{p-\epsilon}\, dt \right)^{\frac{1}{p-\epsilon}} < \infty \right\}, \tag{1.1}$$

where δ is a measurable function in I, have been considered. It has been shown that the interesting case is when δ is left continuous, increasing (i.e. $0 < \epsilon_1 < \epsilon_2 < p-1 \;\Rightarrow\; \delta(\epsilon_1) \le \delta(\epsilon_2)$) such that $\delta(0^+) = 0$ and with values in $]0,1]$.

Let $\mathcal{M}$ be the set of all Lebesgue measurable functions in I with values in $[-\infty, +\infty]$, $\mathcal{M}^+$ the subset of the nonnegative functions, $\mathcal{M}_0$ the subset of the finite a.e. functions, and $\mathcal{M}_0^+$ the subset of the finite a.e., nonnegative functions.

Recently in [2] the following further generalization of $\|f\|_{p),\delta}$ was introduced, where in (1.1) $p - \epsilon$ is changed into a general measurable function.

Definition 1.1 *([2]).* Let $p(\cdot) \in \mathcal{M}$, $p(\cdot) \ge 1$ a.e. and $\delta \in L^\infty(I)$, $\delta > 0$ a.e., $0 < \|\delta\|_\infty \le 1$. The Banach function spaces

$$L^{p[\cdot]),\delta(\cdot)}(I) = \left\{ f \in \mathcal{M}_0 : \|f\|_{p[\cdot]),\delta(\cdot)} = \rho_{p[\cdot]),\delta(\cdot)}(|f|) < \infty \right\}, \tag{1.2}$$

where

$$\rho_{p[\cdot]),\delta(\cdot)}(f) = \operatorname*{ess\,sup}_{x\in I} \rho_{p(x)}\left(\delta(x) f(\cdot)\right) \quad (f \in \mathcal{M}_0^+) \tag{1.3}$$

and

$$\rho_{p(x)}(\delta(x)f(\cdot)) = \begin{cases} \left(\int_I (\delta(x)f(t))^{p(x)} dt\right)^{\frac{1}{p(x)}} & \text{if} \quad 1 \le p(x) < \infty \\ \operatorname*{ess\,sup}_{t\in I}(\delta(x)f(t)) & \text{if} \quad p(x) = \infty \end{cases} \tag{1.4}$$

are called fully measurable grand Lebesgue spaces.

We point out that, in the previous definition, the authors choice the symbol $\rho_{p[\cdot]),\delta(\cdot)}(f)$ with square brackets in $p[\cdot]$ and not the more natural $p(\cdot)$ to avoid confusion since the symbol $p(\cdot)$ is already used in the theory of variable spaces with a different meaning. (See for example the monographs [9,29] for an exhaustive treatment of the variable exponent Lebesgue spaces.)

The (standard) grand Lebesgue spaces $L^{p)}(I)$ can be immediately obtained from (1.4) setting $p(x) = p-x$, $1 < p < \infty$ and $\delta(x) = x$.

The generalized grand Lebesgue spaces (1.1) are evidently included in the spaces (1.2): the function $\delta(\epsilon)^{\frac{1}{p-\epsilon}}$ in (1.1) corresponds to the function $\delta(\epsilon)$ in (1.4).