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Quantum information inequalities via tracial positive linear maps

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ABSTRACT

We present some generalizations of quantum information inequalities involving tracial positive linear maps between C^*-algebras. Among several results, we establish a noncommutative Heisenberg uncertainty relation. More precisely, we show that if $\Phi : \mathcal{A} \to \mathcal{B}$ is a tracial positive linear map between C^*-algebras, $\rho \in \mathcal{A}$ is a Φ-density element and A, B are self-adjoint operators of $\mathcal{A}$ such that $\mathrm{sp}(\text{-i}\rho^{\frac{1}{2}}[A,B]\rho^{\frac{1}{2}}) \subseteq [m, M]$ for some scalers $0 < m < M$, then under some conditions

$$V_{\rho,\Phi}(A)\sharp V_{\rho,\Phi}(B) \geq \frac{1}{2\sqrt{K_{m,M}(\rho[A,B])}}\left|\Phi(\rho[A,B])\right|, \tag{0.1}$$

where $K_{m,M}(\rho[A,B])$ is the Kantorovich constant of the operator $\text{-i}\rho^{\frac{1}{2}}[A,B]\rho^{\frac{1}{2}}$ and $V_{\rho,\Phi}(X)$ is the generalized variance of X. In addition, we use some arguments differing from the scalar theory to present some inequalities related to the generalized correlation and the generalized Wigner–Yanase–Dyson skew information.

1. Introduction and preliminaries

In quantum measurement theory, the classical expectation value of an observable (self-adjoint operator) A in a quantum state (density operator) ρ is expressed by $\mathrm{Tr}(\rho A)$. Also, the classical variance for a quantum state ρ and an observable operator A is defined by $V_\rho(A) := \mathrm{Tr}(\rho A^2) - (\mathrm{Tr}(\rho A))^2$. The Heisenberg uncertainty relation asserts that

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \tag{1.1}$$

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for a quantum state ρ, where $[A,B] := AB - BA$ is the commutator of two observables A and B; see [6]. It gives a fundamental limit for the measurements of incompatible observables. A further strong result was given by Schrödinger [15] as

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}(\mathrm{Cov}_\rho(A,B))|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2, \tag{1.2}$$

where the classical covariance is defined by $\mathrm{Cov}_\rho(A) := \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)$.

Yanagi et al. [18] defined the one-parameter correlation and the one-parameter Wigner–Yanase skew information (is known as the Wigner–Yanase–Dyson skew information; cf. [10]) for operators A, B, respectively, as follows

$$\mathrm{Corr}_\rho^\alpha(A,B) := \mathrm{Tr}(\rho A^*B) - \mathrm{Tr}(\rho^{1-\alpha}A^*\rho^\alpha B) \quad \text{and} \quad I_\rho^\alpha(A) := \mathrm{Corr}_\rho^\alpha(A,A),$$

where $\alpha \in [0,1]$. They showed a trace inequality representing the relation between these two quantities as

$$\left|\mathrm{Re}(\mathrm{Corr}_\rho^\alpha(A,B))\right|^2 \leq I_\rho^\alpha(A)I_\rho^\alpha(B). \tag{1.3}$$

In the case that $\alpha = \frac{1}{2}$, we get the classical notions of the correlation $\mathrm{Corr}_\rho(A,B)$ and the Wigner–Yanase skew information $I_\rho(A)$. The classical Wigner–Yanase skew information represents a for non-commutativity between a quantum state ρ and an observable A.

Luo [11] introduced the quantity $U_\rho(A)$ as a measure of uncertainty by

$$U_\rho(A) = \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_\rho(A))^2}.$$

He then showed a Heisenberg-type uncertainty relation on $U_\rho(A)$ as

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2. \tag{1.4}$$

These inequalities was studied and extended by a number of mathematicians. For further information we refer interested readers to [3,5,8,17].

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with the unit I. If $\mathcal{H} = \mathbb{C}^n$, we identify $\mathbb{B}(\mathbb{C}^n)$ with the matrix algebra of $n \times n$ complex matrices $M_n(\mathbb{C})$. We consider the usual Löwner order $\leq$ on the real space of self-adjoint operators. Throughout the paper, a capital letter means an operator in $\mathbb{B}(\mathcal{H})$. An operator A is said to be strictly positive (denoted by $A > 0$) if it is a positive invertible operator. According to the Gelfand–Naimark–Segal theorem, every C^*-algebra can be regarded as a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. So we may consider elements of $\mathcal{A}$ as Hilbert space operators. We use $\mathcal{A}, \mathcal{B}, \cdots$ to denote C^*-algebras. We denote by $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ the real and imaginary parts of A, respectively. The geometric mean is defined by $A\sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$ for operators $A > 0$ and $B \geq 0$. A W^*-algebra is a $*$-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator. The C^*-algebra of complex valued continuous functions on the compact Hausdorff space Ω is denoted by $C(\Omega)$.

A linear map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ between C^*-algebras is said to be $*$-linear if $\Phi(A^*) = \Phi(A)^*$. It is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is called strictly positive if $A > 0$, then $\Phi(A) > 0$. We say that Φ is unital if $\mathcal{A}, \mathcal{B}$ are unital and Φ preserves the unit. A linear map Φ is called n-positive if the map $\Phi_n : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$ defined by $\Phi_n([a_{ij}]) = [\Phi(a_{ij})]$ is positive, where $M_n(\mathcal{A})$ stands for the C^*-algebra of $n \times n$ matrices with entries in $\mathcal{A}$. A map Φ is said to be completely positive if it is n-positive for every $n \in \mathbb{N}$. According to [16, Theorem 1.2.4] if the range of the positive linear map Φ is commutative, then Φ is completely positive. It is known (see, e.g., [4]) that if Φ is a unital positive linear map, then