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Linear maps preserving determinant of tensor products of Hermitian matrices

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ABSTRACT

Let $m, n \geq 2$ be positive integers and let H_n be the set of $n \times n$ complex Hermitian matrices. We study linear maps $\phi : H_{mn} \to H_{mn}$ satisfying $\det(A \otimes B) = \det(\phi(A \otimes B))$ for all $A \in H_m$, $B \in H_n$. The connection of the problem to quantum information science is mentioned.

1. Introduction

Let M_n be the set of all $n \times n$ complex matrices and let H_n be the set of Hermitian matrices in M_n. The study of linear maps on matrices, operators or other algebraic objects that leave invariant certain functions, subsets or relations is now commonly referred to as the study of linear preserver problems. The first result on linear preservers is due to Frobenius [5] who studied linear maps on matrix algebras preserving the determinant. Frobenius proved that a bijective linear operator $\phi : M_n \to M_n$ satisfies

$$\det(A) = \det(\phi(A)), \qquad A \in M_n,$$

if and only if there exist matrices $M, N \in M_n$ with $\det(MN) = 1$ such that either

$$\phi(A) = MAN, \qquad A \in M_n,$$

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or

$$\phi(A) = MA^tN, \qquad A \in M_n,$$

where A^t denotes the transpose of a matrix $A \in M_n$. More than fifty years later Dieudonné [2] proved that an invertible linear operator $\phi : M_n \to M_n$ maps the set of singular matrices onto itself if and only if there are invertible $M, N \in M_n$ such that ϕ has the above (standard) form. The same is true for linear operators $\phi : M_n \to M_n$ which map the set of invertible matrices onto itself (see [10]). For more results, new directions, and active research (motivated by theory and applications) on linear preserver problems we refer the reader to [7,12] and [1,13].

Recently, many mathematicians have raised questions combining linear preserver problems with quantum information science (see, for example, [3,4,6,8] and the references therein). In quantum physics, quantum states of a system with n physical states are represented as $n \times n$ density matrices in H_n, i.e., positive semi-definite matrices with trace one. Rank one orthogonal projections are pure states. Given two quantum states $A \in H_m$ and $B \in H_n$, their composite state is the tensor (Kronecker) product $A \otimes B \in H_m \otimes H_n \equiv H_{mn}$ (i.e., if $A = [a_{ij}]$, then $A \otimes B = [a_{ij}B]$). Here, let us point out that it is relatively easy to extract information from matrices in tensor product form. For instance, if $A \in M_m$ has eigenvalues $\alpha_1, \dots, \alpha_m$ and $B \in M_n$ has eigenvalues $\beta_1, \dots, \beta_n$, then the eigenvalues of $A \otimes B$ have the form $\alpha_i\beta_j$ with $1 \le i \le m$ and $1 \le j \le n$. Thus, it is interesting to get information on the tensor space $M_m \otimes M_n \equiv M_{mn}$ by examining the properties of the small collection of matrices in tensor form $A \otimes B$. In particular, if we consider a linear map $\phi : M_{mn} \to M_{mn}$ and if one knows the images $\phi(A \otimes B)$ for $A \in M_m$ and $B \in M_n$, then the map ϕ can be completely characterized since every $C \in M_{mn}$ is a linear combination of matrices in tensor form $A \otimes B$. Nevertheless, the challenge is to use the limited information of the linear map ϕ on matrices in tensor form to determine the structure of ϕ.

In [3], Fošner, Huang, Li, and Sze gave a brief survey of recent results on linear preserver problems related to the quantum information science. In addition, they characterized linear operators ϕ on $H_m \otimes H_n$ such that $A \otimes B$ and $\phi(A \otimes B)$ have the same spectrum (resp., spectral radius) for any $A \otimes B \in H_m \otimes H_n$. Motivated by these and other recent results on preservers on tensor states, we consider linear operators ϕ on $H_m \otimes H_n$ such that $A \otimes B$ and $\phi(A \otimes B)$ have the same determinant for any $A \otimes B \in H_m \otimes H_n$.

The paper is organized as follows. In Section 2 we fix the notation and list some basic lemmas which we need in the sequel. The main results are introduced and proved in Section 3. We end this paper with additional remarks and open problems (see Section 4).

2. Preliminary results

Throughout this paper, for a positive number k, I_k denotes the $k \times k$ identity matrix, $E_{ij}^{(k)}$, $1 \le i, j \le k$, denotes the $k \times k$ matrix whose all entries are equal to zero except for the (i, j)-th entry which is equal to 1, and $\mathrm{diag}(a_1, \dots, a_k) \in M_k$ denotes the $k \times k$ diagonal matrix with diagonal entries $a_1, \dots, a_k$. As usual, $A > 0$ (resp., $A < 0$) denotes that $A \in M_k$ is a positive (resp., negative) definite matrix.

Let $m, n \ge 2$ be positive integers. We say that a linear map $\phi : H_{mn} \to H_{mn}$ preserves the determinant of tensor products of matrices if

$$\det(A \otimes B) = \det(\phi(A \otimes B)) \tag{2.1}$$

for all $A \in H_m$ and $B \in H_n$.

Lemma 1. *[11, Theorem 12] Let $C_1, \dots, C_m \in M_{mn}$ be rank-one matrices. If* $\mathrm{rank}\left(\sum_{k=1}^m C_k\right) = mn$, *then there exist invertible matrices $P, Q \in M_{mn}$ such that*

$$C_k = P\left(E_{kk}^{(m)} \otimes I_n\right)Q, \qquad k = 1, \dots, m.$$