J. Math. Anal. Appl. 446 (2017) 1183–1195

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Hypercyclic orbits intersect subspaces in wild ways

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ARTICLE INFO

Article history:
Received 13 July 2016
Available online 21 September 2016
Submitted by R.M. Aron

Keywords:
Hypercyclic operator
Subspace-hypercyclicity
Weak mixing
Mixing
Set of visiting times

ABSTRACT

Let $T : X \to X$ be a hypercyclic operator of an infinite dimensional separable Banach space X. By modifying a construction of Grivaux, we will show that the T-orbit of a hypercyclic vector can intersect certain closed vector subspaces of X in many strange ways. Moreover, the set of visiting times of the orbit to the subspace can also be quite exotic, especially when the operator satisfies a stronger form of hypercyclicity. As a by-product, we will improve and/or supply new proofs to some of the recent results about subspace-hypercyclicity.

1. Introduction

Even though the notion of *hypercyclicity* (the existence of a dense orbit) for bounded linear operators of Banach spaces has been studied extensively in the recent years (see the books [4,8]), our understanding of how a dense orbit intersects a closed vector subspace is still not very clear. An operator T is said to be *subspace-hypercyclic* if some orbit of T intersects a non-trivial closed vector subspace Y in a dense subset of Y. Some of the preliminary investigations pertaining to subspace-hypercyclicity can be found in the recent articles [10,12,9,2]. An example constructed in [9] shows that the intersection of an orbit with a closed vector subspace Y can be somewhere dense in Y without being dense in Y. We remark here that such an example can also be produced using a construction of Grivaux [6], which says that any countable dense linearly independent subset of an infinite dimensional separable Banach space can be realized as an orbit of a hypercyclic operator.

We will modify the construction of Grivaux to show that the orbit of a hypercyclic vector can intersect certain closed vector subspaces in many strange ways. Moreover, the set of visiting times of the orbit to the subspace can also be quite exotic, especially when the operator satisfies a stronger form of hypercyclicity. As a by-product of our results, we will improve and/or supply new proofs to some of the recent results about subspace-hypercyclicity.

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http://dx.doi.org/10.1016/j.jmaa.2016.09.036

A lesson that we learn from these results is that certain global properties of hypercyclic operators may fail when restricted to a closed vector subspace.

The paper is organized as follows. In the next section, we prove our main theorem and also establish a preparatory lemma that is needed for later applications. Several applications of our main theorem to hypercyclic operators are discussed in the third section. Many typical applications to operators satisfying stronger forms of hypercyclicity are presented in the fourth section. Two major results in this article – Theorem 2 and Theorem 9 – are stated in great generality and hence are a bit abstract and technical. But otherwise, we have tried to keep the statements of the results as simple as possible, avoiding largely the temptation to go for the most general statement. Therefore, our list of applications is by no means exhaustive. By going through the results, it will be evident to the reader that a few more variants and generalizations of the results that we establish can also be proved by the same technique.

Definitions and notations will be presented as and when required.

2. Main theorem and a preparatory lemma

If X is a Banach space, let X^* denote its dual space consisting of all bounded linear functionals on X, and $Lat(X)$ denote the lattice of all closed vector subspaces of X. Also, let

$\mathcal{L}(X) = \{T : X \to X : T \text{ is bounded linear}\}$, and

$\mathcal{I}(X) = \{T \in \mathcal{L}(X) : T \text{ is invertible in } \mathcal{L}(X)\}$.

The construction of Grivaux [6] involves a repeated application of essentially the following observation.

Proposition 1. *Let X be an infinite dimensional Banach space, $Y \in Lat(X)$, $w \in X \setminus Y$, and $Z \subset X$ be a dense subset. Then for every $R \in \mathcal{I}(X)$ and every $\varepsilon > 0$, there is $S \in \mathcal{I}(X)$ such that $S|_Y = R|_Y$, $Sw \in Z$, and* $\max\{\|R - S\|, \|R^{-1} - S^{-1}\|\} < \varepsilon$.

Proof. We provide a proof for the sake of completeness. Since $\mathcal{I}(X)$ is open in $\mathcal{L}(X)$, there is $\beta \in (0, \varepsilon)$ such that $S \in \mathcal{I}(X)$ for every $S \in \mathcal{L}(X)$ with $\|R - S\| < \beta$. Since $S \mapsto S^{-1}$ is a continuous map of $\mathcal{I}(X)$, there is $\delta \in (0, \beta)$ such that $\|R^{-1} - S^{-1}\| < \varepsilon$ for every $S \in \mathcal{I}(X)$ with $\|R - S\| < \delta$. Choose $z \in Z$ with $\|z - Rw\| < \delta\|w\|$, and then choose $\phi \in X^*$ such that $\phi|_Y \equiv 0$ and $\phi(w) = 1 = \|\phi\|\|w\|$. Define $S : X \to X$ as $Sx = Rx + \phi(x)(z - Rw)$. Clearly, $S \in \mathcal{L}(X)$, $S|_Y = R|_Y$ and $Sw = z \in Z$. Moreover, $\|R - S\| \leq \|\phi\|\|z - Rw\| < \delta\|\phi\|\|w\| = \delta$. Hence $S \in \mathcal{I}(X)$ and $\max\{\|R - S\|, \|R^{-1} - S^{-1}\|\} < \varepsilon$ by the choice of δ. □

We modify the construction in [6] to prove our main theorem stated below. We will show that given two countable collections $\{A_j\}$, $\{C_j\}$ of countable dense subsets of an infinite dimensional separable Banach space, we can find an invertible operator R arbitrarily close to the identity operator such that $R(A_j) = C_j$ for each j provided some mild conditions are satisfied.

It will be convenient to use the notation $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Theorem 1. *Let X be an infinite dimensional separable Banach space, and J be a countable indexing set. For each $j \in J$, let $A_j = \{a_{j,n} : n \in \mathbb{N}_0\}$ and $C_j = \{c_{j,n} : n \in \mathbb{N}_0\}$ be countably infinite dense subsets of X such that*

(1) $A_i \cap A_j = \emptyset = C_i \cap C_j$ for any two distinct $i, j \in J$, and

(2) both $\bigcup_{j \in J} A_j$ and $\bigcup_{j \in J} C_j$ are linearly independent subsets of X.

Then for every $\varepsilon > 0$, there is $R \in \mathcal{I}(X)$ with $\|I - R\| < \varepsilon$ such that $R(A_j) = C_j$ for every $j \in J$. Moreover, if for a fixed index $j \in J$, we have $\|a_{j,0} - c_{j,0}\| < \varepsilon\|a_{j,0}\|$, then R may be chosen with the extra property $Ra_{j,0} = c_{j,0}$ for that particular j.