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Steinhaus' lattice-point problem for Banach spaces ☆

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ABSTRACT

Steinhaus proved that given a positive integer n, one may find a circle surrounding exactly n points of the integer lattice. This statement has been recently extended to Hilbert spaces by Zwoleński, who replaced the integer lattice by any infinite set that intersects every ball in at most finitely many points. We investigate Banach spaces satisfying this property, which we call (S), and characterise them by means of a new geometric property of the unit sphere which allows us to show, *e.g.*, that all strictly convex norms have (S), nonetheless, there are plenty of non-strictly convex norms satisfying (S). We also study the corresponding renorming problem.

1. Introduction and statement of the main results

The following feature of the integer lattice in the Euclidean plane was probably first observed by Steinhaus [10, Problem 24 on p. 17]: for any natural number n one may find a circle surrounding exactly n lattice points. Zwoleński [11] generalised this fact to the setting of Hilbert spaces in the following manner. He replaced the set of lattice points by a more general *quasi-finite* set, *i.e.*, an infinite subset A of a metric space X such that each ball in X contains only finitely many elements of A. His result then reads as follows.

Theorem ([11]). *Let A be a quasi-finite subset of a Hilbert space X. Then there exists a dense subset $Y \subset X$ such that for every $y \in Y$ and $n \in \mathbb{N}$ there exists a ball B centred at y with $|A \cap B| = n$.*

Let us then distill the property that we will term *Steinhaus' property* (S). A metric space X has this property if, by definition,

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(S) for any quasi-finite set $A \subset X$ there exists a dense set $Y \subset X$ such that for all $y \in Y$ and $n \in \mathbb{N}$ there exists a ball B centred at y with $|A \cap B| = n$.

We translate condition (S), formulated above, into three equivalent statements concerning the geometry of the unit ball of a Banach space. Roughly speaking, they require that, locally, the unit sphere of X does not look the same at any two distinct points. This approach will be particularly beneficial, as it will allow us to identify spaces that share that property with Hilbert spaces, yet of a very different nature. Our first main result then reads as follows.

Theorem A. *Let X be a Banach space. The following assertions are equivalent:*

(S) *X has Steinhaus' property;*
(S_1) *for any quasi-finite set $A \subset X$ there exists a dense set $Y \subset X$ such that for every $y \in Y$ there exists a ball B centred at y with $|A \cap B| = 1$;*
(S') *for all $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$ and each $\delta > 0$ there exists a $z \in X$ with $\|z\| < \delta$ such that one of the vectors $x + z$ and $y + z$ has norm greater than 1, whereas the other has norm smaller than 1;*
(S") *for all $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$ and each $\delta > 0$ there exists a $z \in X$ with $\|z\| < \delta$ such that $\|x + z\| \neq \|y + z\|$.*

In other words, condition (S") means exactly that one cannot find a 'neighbourhood' of parallel line segments on the unit sphere of equal length. This seems to be a new geometric property which, as we will see, is essentially weaker than strict convexity. Notice that, in contrast to many other classical properties, property (S) is not inherited by subspaces and, in a sense, is neither local nor global.

Properties (S') and (S") are related to another (weaker) property of 'non-flatness' of the unit sphere:

(F) the unit sphere S_X of X does not contain any flat faces, that is to say, there is no non-empty subset of S_X, open in the relative norm topology, that is contained in a hyperplane.

Here by a *hyperplane* of X we understand a translation of a subspace of X of codimension 1, *i.e.*, a set of the form $x + \ker(x^*)$ for some $x \in X$ and $x^* \in X^*$. Note, however, that (F) does not imply (S") that is witnessed by the norm $\|(x, y, z)\| = \max\{\sqrt{x^2 + y^2}, |z|\}$ for $(x, y, z) \in \mathbb{R}^3$ (consider the points $(1, 0, 0)$ and $(1, 0, \frac{1}{2})$). However, whether every Banach space admits a renorming satisfying (F) seems to be an attractive open problem.

We employ the announced equivalence to extend Zwoleński's result to strictly convex Banach spaces (Corollary 2). It is well-known that not every Banach space admits a strictly convex renorming, just to mention the examples of $\ell_\infty(\Gamma)$ for any uncountable set Γ (see [3] and [4, §4.5]) or the quotient space ℓ_∞/c_0 ([2]). This motivates the question of whether strict convexity and property (S) are equivalent at the level of renormings, and a negative answer is a part of our next result.

Theorem B. *Assuming that the continuum is a real-valued measurable cardinal, there exists a non-strictly convexifable Banach space whose norm satisfies (S). Moreover, for any Banach space X we have:*

(i) *if $\dim X \leqslant 2$, then X has property (S) if and only if X is strictly convex;*
(ii) *if $\dim X > 2$ and X admits a renorming with property (S), then it also admits a non-strictly convex renorming with property (S).*