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Kelvin–Voight equation with p-Laplacian and damping term: Existence, uniqueness and blow-up

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ABSTRACT

In this paper, we consider the initial-boundary value problem for a generalized Kelvin–Voight equation with p-Laplacian and a damping term:

$$\vec{v}_t + (\vec{v}\cdot\nabla)\,\vec{v} + \nabla\mathbb{P}(x,t) = \varkappa\,\mathrm{div}\,(\nabla\vec{v}_t) + \nu\,\mathrm{div}\left(|\nabla\vec{v}|^{p-2}\,\nabla\vec{v}\right) + \gamma\,|\vec{v}|^{m-2}\,\vec{v},$$

$$\mathrm{div}\,\vec{v} = 0.$$

Here $\vec{v}(x,t)$ is the velocity field, $\mathbb{P}(x,t)$ is the pressure, ν is the viscosity kinematic coefficient, and $\varkappa$ is the viscosity relaxation coefficient (is a length scale parameter characterizing the elasticity of the fluid). The coefficient γ and the exponents p, m are given constants. Under appropriate conditions on the data, we prove the existence and uniqueness of the global and local weak solutions. Under several assumptions on the exponents p, m, the coefficients ν, $\varkappa$, and specified initial data, a finite time blow-up and the behavior of the solutions for large times are also established.

1. Introduction

Let $\Omega\subset\mathbb{R}^n$, $n\geq 2$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$ and $Q_T=\Omega\times(0,T)$, $\Gamma_T=\partial\Omega\times(0,T)$, $T<\infty$. We study the following initial-boundary value problem for the generalized Kelvin–Voight equation with p-Laplacian and damping term:

$$\vec{v}_t + (\vec{v}\cdot\nabla)\,\vec{v} + \nabla\mathbb{P} = \varkappa\,\mathrm{div}\,(\nabla\vec{v}_t) + \nu\,\mathrm{div}\left(|\nabla\vec{v}|^{p-2}\,\nabla\vec{v}\right) + \gamma\,|\vec{v}|^{m-2}\,\vec{v}, \tag{1.1}$$

$$\mathrm{div}\,\vec{v} = 0 \text{ in } Q_T, \tag{1.2}$$

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$$\vec{v}(x,0) = \vec{v}_0(x) \ x \in \Omega, \tag{1.3}$$

$$\vec{v}(x,t) = 0 \text{ on } \Gamma_T. \tag{1.4}$$

Here $\vec{v}(x,t)$ is the velocity field, $\mathbb{P}(x,t)$ is the pressure, ν is the viscosity kinematic coefficient, and $\varkappa$ is the viscosity relaxation coefficient (is a length scale parameter characterizing the elasticity of the fluid). The coefficient γ and the exponents p, m are given constants.

The damping term $\gamma|\vec{v}|^{m-2}\vec{v}$, for $\gamma < 0$, is usually called *absorption term* and *nonlinear source term* when $\gamma > 0$. For the motivation and physical justification of the importance of this term, we refer to Antontsev and Oliveira [3], where a modified Navier–Stokes problem was studied.

The global existence and uniqueness of weak solutions to the original Kelvin–Voight problems, i.e. system (1.1)–(1.4) with $p = 2$ and $\gamma = 0$, were studied by Oskolkov in [18–20]. Equations like (1.1) appear in the mathematical modeling of various physical phenomena, e.g., in the analysis of a viscous flow in media with memory, the non-equilibrium water–oil displacement in porous media, the theory of fluid flow with delay, the theory of non-Newtonian flows, and in many other problems (see, e.g., [18,19,21,7,8,25] and references therein).

In the case $\varkappa = 0$ and $\gamma > 0$, problem (1.1)–(1.4) reduces to the so called *generalized Navier–Stokes problem with p-Laplacian*. The first results on the existence of weak solution to these problems were obtained in [13] and [16]. In the last years, we observe an increasing interest to modified Navier–Stokes problems, as well as parabolic, and hyperbolic equations. The authors prove the existence results and establish various properties of weak solutions to these equations (see, e.g., [4,2,6,11,23,24], and the further references therein).

In [3,5], the existence, uniqueness (in the case $n = 2$), the extinction in a finite time, and large time behavior properties of weak solution to the generalized Navier–Stokes problem were proved. However, in these papers the existence of weak solution remains an open question if $1 < p \leq \frac{2n}{n+2}$.

Equation (1.1) belongs to the class of pseudoparabolic (or Sobolev type) equations. The analysis of blow up behavior of solutions to equations of this type was done e.g., in [1,9,17] (see also the reference therein). However, in these papers the existence and uniqueness results were not obtained. The existence of a weak solution to a pseudoparabolic equation without the damping term was proved by Liu in [10].

The outline of the paper is as follows. In Section 1, we formulate the problem and give a review of the results related to this problem. In Section 2, we introduce the notation, formulate some necessary preliminaries, and define the weak solution to problem (1.1)–(1.4). In Section 3 we establish a global and local existence of weak solutions to problem (1.1)–(1.4). In Section 4, we discuss the uniqueness of weak solution to this problem. In Section 5, we establish a blow-up result for problem (1.1)–(1.4) in the case $\gamma > 0$. Finally, in the last section we prove the large time behavior of weak solution to the problem (1.1)–(1.4) in the case $\gamma < 0$. It should be noted that the condition $\varkappa > 0$ is basic at the proof of results of article. These results not to be extended to the case $\varkappa = 0$.

2. Preliminaries

2.1. Functional spaces

The goal of the section is to introduce the necessary notation, the definition of weak solution to problem (1.1)–(1.4), and an important lemma which is used below in this paper.

Following [16,22] we define the functional spaces used throughout the paper and describe briefly their properties. Namely, the norm of $L_p(\Omega)$ is denoted by $\|u\|_p = \left(\int\limits_\Omega |u|^p dx \right)^{1/p}$. Let us also introduce the following functional spaces widely used in the Mathematical Fluid Mechanics: