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Projection operators nearly orthogonal to their symmetries ☆

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ABSTRACT

For any order 2 automorphism α of a C*-algebra A (a symmetry of A), we prove that for each projection e such that $\|e\alpha(e)\| \leq \frac{9}{20}$, there exists a projection q with $q\alpha(q) = 0$ satisfying the norm estimate

$$\|e - q\| \leq \frac{1}{2}\|e\alpha(e)\| + 4\|e\alpha(e)\|^2.$$

In other words, if e is a projection that is "nearly orthogonal" to its symmetry $\alpha(e)$ in the sense that the norm $\|e\alpha(e)\|$ is no more than $\frac{9}{20}$, then e can be approximated by a projection q that is exactly orthogonal to its symmetry in a fairly optimal fashion. (Optimal in the sense that the first term in the estimate satisfies $\frac{1}{2}\|e\alpha(e)\| \leq \|e-q\|$ for any such q.)

1. Introduction

The purpose of this paper is to obtain a fine estimate for the norm difference $\|e - q\|$ in terms of the norm $\|e\alpha(e)\|$ of a projection e relative to a symmetry α (order 2 automorphism), where q is a projection that is orthogonal to its symmetry (i.e. $q\alpha(q) = 0$). The norm $\|e\alpha(e)\|$ measures the degree to which e is or is not orthogonal to its symmetric image $\alpha(e)$. It is shown that for all C*-algebras this degree does not have to be too small in order that the projection e can be approximated by a projection q that is exactly orthogonal to its symmetry. We show the existence for such fine approximation when the norm $\|e\alpha(e)\|$ is at most $\frac{9}{20} = 0.45$. Further, a bound for the norm $\|e - q\|$ is expressed in terms of a simple quadratic function of $\|e\alpha(e)\|$. The main result is the following.

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Theorem 1.1. *Let A be any C*-algebra and α a symmetry of A. If e is a projection in A such that $\|e\alpha(e)\| < \xi(\approx 0.455)$, then there exists a projection q in the C*-subalgebra generated by $e, \alpha(e)$ such that*

$$q\alpha(q) = 0, \qquad \|e - q\| \le \tfrac{1}{2}\|e\alpha(e)\| + 4\|e\alpha(e)\|^2. \tag{1.1}$$

Theorem 1.2. *Let e be any projection operator and u any Hermitian unitary operator on Hilbert space such that $\|eue\| < \xi(\approx 0.455)$. Then there exists a projection operator q such that*

$$quq = 0, \qquad \|e - q\| \le \tfrac{1}{2}\|eue\| + 4\|eue\|^2.$$

Further, q is in the C-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by e, ueu^*.*

The number $\xi \approx 0.4550898$ is the positive root of $x^2(2+4F(x^2)) = 1$ (where F is defined by (2.1) below). It is clear that Theorem 1.2 follows from 1.1 (since the symmetry on $\mathcal{B}(\mathcal{H})$ in this case is $\alpha(x) = uxu^*$).

The precision of the inequality (1.1) is recognized by noting that the norm $\|e - q\|$ is always at least the first term on the right side:

$$\tfrac{1}{2}\|e\alpha(e)\| \le \|e - q\| \tag{1.2}$$

for any projection q that is orthogonal to its symmetry ($q\alpha(q) = 0$). Indeed, this is easy to see from the equality

$$e\alpha(e) = (e - q)\alpha(q) + e\alpha(e - q)$$

which gives (1.2). The theorem therefore estimates the norm $\|e-q\|$ from its minimum value (over such q's) to within a quadratic order of magnitude:

$$\tfrac{1}{2}\|e\alpha(e)\| \ \le \ \|e - q\| \ \le \ \tfrac{1}{2}\|e\alpha(e)\| + 4\|e\alpha(e)\|^2.$$

In order to improve our estimates, we used the following anticommutator norm formula that we proved in [2].

Theorem 1.3. *(See [2].) For any two projection operators f, g on Hilbert space, one has*

$$\|fg + gf\| = \|fg\| + \|fg\|^2.$$

We note that a C*-algebra A that possesses a symmetry contains non-trivial α-orthogonal positive elements. For example, pick a Hermitian element h such that $\alpha(h) \neq h$ and let $x = h - \alpha(h)$, a nonzero Hermitian element such that $\alpha(x) = -x$. The positive part $a = \frac{1}{2}(|x| + x)$ of x is non-zero (since the spectrum of x contains positive and negative real numbers) and clearly satisfies $a\alpha(a) = 0$. If further, the hereditary C*-subalgebra generated by a, namely $\overline{aAa}$, contains projections then these will automatically be α-orthogonal projections. In particular, if A has real rank zero[1] and has a symmetry, then it contains many α-orthogonal projections.

Theorem 1.1 can be applied in particular to the flip automorphism $U \to U^{-1}, V \to V^{-1}$ of the rotation C*-algebra A_θ generated by unitaries U, V subject to the commutation relation $VU = e^{2\pi i\theta}UV$ – or, indeed, to the flip on any higher dimensional noncommutative torus. The result can also be applied to the noncommutative Fourier transform $U \to V \to U^{-1}$ restricted the fixed point subalgebra of A_θ under the flip.

[1] That is, each Hermitian element can be approximated by a Hermitian with finite spectrum.