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1-Grothendieck $C(K)$ spaces ☆

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ABSTRACT

A Banach space is said to be Grothendieck if weak and weak* convergent sequences in the dual space coincide. This notion has been quantified by H. Bendová. She has proved that ℓ_∞ has the quantitative Grothendieck property, namely, it is 1-Grothendieck. Our aim is to show that Banach spaces from a certain wider class are 1-Grothendieck, precisely, $C(K)$ is 1-Grothendieck provided K is a totally disconnected compact space such that its algebra of clopen subsets has the so called Subsequential completeness property.

1. Introduction and main results

We say that a Banach space X is *Grothendieck* if each weak* convergent sequence in the dual space X^* is necessarily weakly convergent. Naturally, every reflexive space is Grothendieck. Classical example of a nonreflexive Grothendieck space is ℓ_∞ due to Grothendieck [3]. More generally, $C(K)$ is Grothendieck if K is a compact Hausdorff F-space (i.e., disjoint open F_σ subsets of K have disjoint closures) [8]. According to R. Haydon [4, 1B Proposition], $C(K)$ is Grothendieck provided K is a totally disconnected compact space such that its algebra of clopen subsets has the so called Subsequential completeness property. In [4] Haydon has constructed such a space which moreover does not contain isomorphic copy of ℓ_∞. In [7] H. Pfitzner has shown that each von Neumann algebra is a Grothendieck space. Some other Grothendieck spaces are the Hardy space H^∞ [2] or weak L^p spaces [6].

The Grothendieck property has been quantified by H. Bendová in [1] as follows:

Definition 1.1 *(The Quantitative Grothendieck property).* Let X be a Banach space. For a bounded sequence $(x_n^*)_{n\in\mathbb{N}}$ in the dual X^* define two moduli:

$$\delta_{w^*}(x_n^*) := \sup\{\operatorname{diam}\operatorname{clust}(x_n^*(x)) :\ x \in B_X\},$$

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$$\delta_w(x_n^*) := \sup\{\operatorname{diam}\operatorname{clust}(x^{**}(x_n^*)) : \ x^{**} \in B_{X^{**}}\},$$

where $\operatorname{clust}(a_n)$ with (a_n) being a sequence denotes the set of all cluster points of (a_n). Let $c \geq 1$. We say that X is *c-Grothendieck* if $\delta_w(x_n^*) \leq c\delta_{w^*}(x_n^*)$ whenever $(x_n^*)_{n\in\mathbb{N}}$ is a bounded sequence in X^*.

It is known that ℓ_∞ is even 1-Grothendieck due to H. Bendová [1, Theorem 1.1]. We generalize this result on a wider class of spaces. This class also includes the space which Haydon has constructed [4].

Now, let us remind the definitions of the above mentioned notions which were essential for Haydon's construction.

Definition 1.2.

(1) We say that a topological space T is totally disconnected if it contains at least two different points and each two different points are separated by a clopen set.
(2) We say that a totally disconnected compact space K is a *Haydon space* if the algebra of its clopen subsets has *the Subsequential completeness property* (SCP), i.e., if for any sequence $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint clopen sets there is an infinite set $M \subset \mathbb{N}$ such that the union of $(U_m)_{m\in\mathbb{M}}$ has open closure.

Our aim is to show that $C(K)$, that is $C(K;\mathbb{R})$ or $C(K;\mathbb{C})$, has the Quantitative Grothendieck property, namely it is 1-Grothendieck, provided K is a Haydon space. Since the Quantitative Grothendieck property implies the Qualitative one, our result strengthens Haydon's proposition [4, 1B Proposition].

Theorem 1.3. *If S is a Haydon space then $C(S)$ is 1-Grothendieck.*

The proof of the theorem is in section 3. Since 1-Grothendieck property of $C(K;\mathbb{R})$ and $C(K;\mathbb{C})$ being equivalent it suffices to get our result for real case. The equivalence is proved in section 2.

Corollary 1.4. *$C(K)$ is 1-Grothendieck whenever K is a σ-Stonean compact Hausdorff space (i.e., a compact Hausdorff space in which the closure of any open F_σ set is open). In particular, $C(K)$ is 1-Grothendieck whenever K is an extremally disconnected (i.e., every open set has open closure) compact Hausdorff space.*

Proof. In view of [8, Theorem A] every σ-Stonean compact Hausdorff space is Haydon. □

Corollary 1.5. *There is a nonreflexive 1-Grothendieck space not containing ℓ_∞.*

Proof. As we have already said Haydon had constructed a Haydon space K with $C(K)$ not containing ℓ_∞ [4]. □

2. Real and complex case equivalence

This section is devoted to the following proposition.

Proposition 2.1. *Let K be a compact Hausdorff space. Then the following assertions are equivalent:*

(i) $C(K;\mathbb{R})$ is 1-Grothendieck.
(ii) $C(K;\mathbb{C})$ is 1-Grothendieck.
(iii) Whenever μ_n and ν_n, $n \in \mathbb{N}$, are two sequences of Radon probability measures on K such that μ_m and ν_n are mutually singular for each $m, n \in \mathbb{N}$ and $\varepsilon > 0$ then there are $\Lambda \subset \mathbb{N}$ infinite and disjoint compact sets $A, B \subset T$ such that for each $n \in \Lambda$ we have $\mu_n(A) > 1 - \varepsilon$ and $\nu_n(B) > 1 - \varepsilon$.