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Topological structure of a class of planar self-affine sets ☆

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ABSTRACT

Let $T = T(A, \mathcal{D}^*)$ be a disk-like $\mathbb{Z}^2$-tile generated by an expanding 2×2 matrix A and a digit set $\mathcal{D}^* \subset \mathbb{Z}^2$. We study the subset F of T defined by $AF = F + \mathcal{D}$, where $\mathcal{D} \subsetneq \mathcal{D}^*$ is a sub-digit set. By studying a periodic extension $H = F + \mathbb{Z}^2$, we classify F into three types according to their topological properties, which generalizes a result of Lau et al. [13]. We also provide some simple criteria for such classification.

1. Introduction

Let A be a $d \times d$ expanding matrix (all its eigenvalues have moduli > 1) with integer entries. It is well known [9,10] that for any finite set $\mathcal{D} \subset \mathbb{R}^d$, there exists a unique non-empty compact set $T := T(A, \mathcal{D})$ such that

$$T = \bigcup_{d \in \mathcal{D}} A^{-1}(T + d) = \left\{ \sum_{j=1}^{\infty} A^{-j} d_j; \quad d_j \in \mathcal{D} \right\}. \tag{1.1}$$

The set T is called a *self-affine set* [8].

If the cardinality $\# \, \mathcal{D} = |\det(A)|$ and T has non-void interior, then T is called a *self-affine tile*; in this case, it is well known that T can tile $\mathbb{R}^d$ translationally [2,10].

Recently, the topological and metric properties of a special kind of self-affine sets have been studied by many works. A self-affine set defined in (1.1), which we denote by F now, is called a *fractal square*, if

$$A = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \text{ and } \ \mathcal{D} \subsetneq \{0, 1, 2, \ldots, n-1\}^2 \ \text{ with } \ n \geq 2.$$

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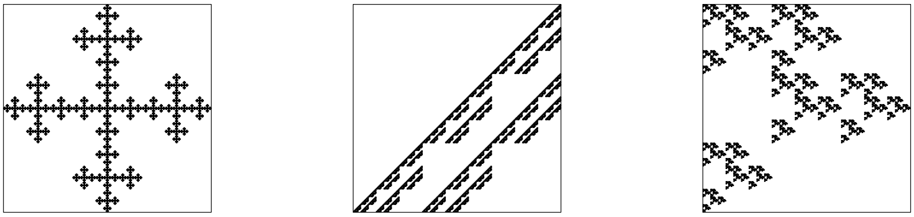

Fig. 1. The pictures indicate the three types of topology of fractal squares (see [13]).

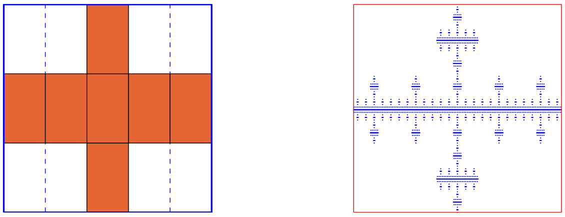

Fig. 2. The self-affine carpets of McMullen.

Equivalently, F is the compact set satisfying

$$F = (F + \mathcal{D})/n. \tag{1.2}$$

Motivated by the study of Lipschitz equivalence of fractal squares, Xi and Xiong [17] and Roinestad [16] gave characterization for totally disconnectedness of fractal squares. Recently, Lau, Luo and Rao [13] provided a more complete understanding on the topological structure of fractal square through their components, see Fig. 1. For brevity, we use *component* to mean *connected component*, and a *non-trivial component* means it has more than one point. The main idea of [13] is to study the topology of a periodic extension $H = F + \mathbb{Z}^2$. They show that

Theorem 1.1. *([13]) Let F be a fractal square as in (1.2). Then*

(i) If H^c (the complement of H) has a bounded component, then F contains a non-trivial component that is not a line segment; in this case, all components of H^c are bounded.
(ii) If H^c has an unbounded component, then F is either totally disconnected, or all non-trivial components of F are parallel line segments.

In this paper, we show the phenomena of Theorem 1.1 hold in a more general setting.

First, we replace the unit square by an integral self-affine tile T which is a disk-like $\mathbb{Z}^2$-tile. Precisely, let A be an expanding 2×2 matrix with integer entries, $\mathcal{D}^* \subset \mathbb{Z}^2$, and $T = T(A, \mathcal{D}^*)$ be a self-affine tile on the plane defined by (1.1); we assume that

(i) T is disk-like, that is, T is homeomorphic to a closed disk.
(ii) T is a $\mathbb{Z}^2$-tile, that is, $T + \mathbb{Z}^2$ is a tiling of $\mathbb{R}^2$.

Let $\mathcal{D} \subsetneq \mathcal{D}^*$. We study the topological structure of fractal F satisfying

$$F = \bigcup_{d \in \mathcal{D}} A^{-1}(F + d). \tag{1.3}$$

The self-affine set in (1.3) is a much larger class than the fractal square. See Figs. 2, 3, 4. Since the 'mother' tile of F has nice topological and algebraic properties, the fractal F also has nice properties as we will show: