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Minimal conditions for implications of Gronwall–Bellman type

Martin Herdegen [a,1], Sebastian Herrmann [b,*,2]

[a] *Department of Statistics, University of Warwick, Coventry, CV4 7AL, UK*
[b] *Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA*

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ABSTRACT

Gronwall–Bellman type inequalities entail the following implication: if a sufficiently integrable function satisfies a certain homogeneous linear integral inequality, then it is nonpositive. We present a minimal (necessary and sufficient) condition on the Borel measure underlying the integrals for this implication to hold. The condition is also a necessary prerequisite for any nontrivial bound on solutions to inhomogeneous linear integral inequalities of Gronwall–Bellman type.

1. Introduction

It is the purpose of this paper to characterise all Borel measures μ on $\mathbb{R}$ for which the following implication holds:

$$y(t) \leq \int_{(-\infty,t)} y \,\mathrm{d}\mu \text{ for } \mu\text{-a.e. } t \in \mathbb{R} \implies y(t) \leq 0 \text{ for } \mu\text{-a.e. } t \in \mathbb{R}. \tag{1.1}$$

Here, y is any Borel function on $\mathbb{R}$ with $\int_{(-\infty,t)} |y| \,\mathrm{d}\mu < \infty$ for each $t \in \mathbb{R}$, so that the integrals in (1.1) exist.

Does the implication (1.1) hold for all Borel measures? If μ is the restriction of the Lebesgue measure to an interval of the form (c, d) for $-\infty < c < d \leq \infty$, then the implication (1.1) holds and is a special case of the famous Gronwall–Bellman lemma (see, e.g., [19, Lemma D.2]). However, if $\mu(\mathrm{d}t) = \frac{1}{t-c}\mathbf{1}_{(c,d)}(t)\,\mathrm{d}t$, then the function $y(t) = t - c$ satisfies the integral inequality in (1.1), but $y > 0$ μ-a.e., so that (1.1) fails in this case.

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* Corresponding author.
E-mail addresses: M.Herdegen@warwick.ac.uk (M. Herdegen), sherrma@umich.edu (S. Herrmann).
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It turns out that the decisive difference between these two examples is that in the former, $\mu((c,t)) < \infty$ for some $t > c$, while in the latter, $\mu((c,t)) = \infty$ for every $t > c$. Indeed, our main result Theorem 2.5 entails that the implication (1.1) holds if and only if the following condition on the measure μ holds:

(M) For each $a \in [-\infty, \infty)$, there is $t > a$ such that $\mu_{\mathrm{sf}}((a,t)) < \infty$.

Here, μ_{sf} is the so-called semi-finite part of μ. (If μ is semi-finite as in the two examples above, then $\mu_{\mathrm{sf}} = \mu$; cf. Proposition 2.1.)

In addition to this "global" result, we also provide a "local" version, which states that for fixed $a \in [-\infty, \infty)$, the "localised" implication

$$\exists b > a : \left[y(t) \leq \int_{(a,t)} y \, \mathrm{d}\mu \text{ for } \mu\text{-a.e. } t \in (a,b) \implies y(t) \leq 0 \text{ for } \mu\text{-a.e. } t \in (a,b) \right] \tag{1.2}$$

is equivalent to the existence of $t > a$ such that the semi-finite part of μ puts finite mass on (a,t). A consequence of our main result is that condition (M) is also necessary for any nontrivial bound on solutions to *inhomogeneous* linear integral inequalities; see the last paragraph of Section 2.

Bounds on solutions to integral or differential inequalities are an important tool for the analysis of various integral or differential equations.[3] The classic results of Gronwall [7], Reid [17], and Bellman [2] have been extended in many different ways over the past century; we refer to [3] and to the monograph [1] for an overview and an extensive list of references. While most of the extant results stay within the realm of ordinary Riemann integration, also other integrals are considered in the literature: Riemann–Stieltjes integrals [13,5], modified Stieltjes integrals [18], abstract Stieltjes integrals [9,14], Lebesgue–Stieltjes integrals [15,16,4], and integrals on general measure spaces. In particular, very general results on Gronwall–Bellmann type inequalities for general measure spaces were obtained by Horváth [10–12] (see also Győri and Horváth [8]). However, in the special case of the homogeneous linear integral inequality considered in (1.1), our condition (M) is still weaker than the conditions imposed in [10, Theorem 3.1].

The remainder of the article is organised as follows. Section 2 states and discusses our main results. Section 3 contains auxiliary results. The proofs of our main results are in Section 4.

2. Main results and ramifications

Before we can state our main results, we need to introduce the so-called semi-finite part of a measure.

Semi-finite measures Fix a measure space (X, Σ, μ). Recall that μ is called *semi-finite* if for every $E \in \Sigma$ with $\mu(E) = \infty$, there is $F \in \Sigma$ such that $F \subset E$ and $0 < \mu(F) < \infty$ [6, Definition 211F]. As in [6, Exercise 213X (c)], the *semi-finite part* of μ is the measure $\mu_{\mathrm{sf}} : \Sigma \to [0, \infty]$ given by

$$\mu_{\mathrm{sf}}(E) = \sup\{\mu(E \cap F) : F \in \Sigma, \, \mu(F) < \infty\}.$$

The following proposition collects basic facts about μ_{sf}. We omit the proofs (see [6, Lemma 213A and Exercise 213X (c)]).

[3] For instance, using standard arguments (see, e.g., [1, Section I.3.1] for the classic case or [16, Section 4] for the case of Lebesgue–Stieltjes integrals), our main result can be used to derive a uniqueness result for integral equations with Borel measures satisfying condition (M).