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A uniqueness result for a class of non-strictly convex variational problems ☆

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ABSTRACT

Let Ω be a smooth domain in $\mathbb{R}^2$, we prove that if $g\colon [0,+\infty) \to [0,+\infty]$ is convex with $g(0) < g(t)$ whenever $t > 0$ then there exists an unique minimizer $u \in C^{0,1}(\Omega)$ of the functional $u \mapsto \int_\Omega g(|\nabla u|)\, dxdy$ among all Lipschitz-continuous functions that assume the same value of u on $\partial\Omega$.

1. Introduction

Let us consider an integral of the Calculus of Variations

$$F(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$, u is a real function defined on Ω in a Sobolev space, say $W^{1,p}(\Omega)$, and $f(x, s, \xi)$ is a Caratheodory function, i.e. measurable in x and continuous in s, ξ. The study of the existence of minimizers of F in a Dirichlet class $u \in u_0 + W^{1,p}(\Omega)$ via Direct Methods is based on the (sequential) lower semicontinuity of F in the weak topology of $W^{1,p}(\Omega)$. It is well known, starting by the classical work of Tonelli, that the lower semicontinuity of F is linked to the convexity of the integrand f with respect to the variable ξ. However, for integrand function not strictly convex uniqueness is not guaranteed.

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In this paper we are interested to uniqueness of minimizers of functionals of the form

$$G(u) = \int_{\Omega} g(|\nabla u(x)|)\, dx \tag{1.2}$$

with suitable prescribed boundary conditions, when g is convex but not necessarily strictly convex. The problem of uniqueness of minimizers for non-strictly convex functionals (1.2) appears when one deals with a non-convex problem and applies the relaxation methods. In fact, the existence or not existence for non-convex integrals is related to the non-uniqueness of minimizers of the (not strictly) convexified problem. The mathematical literature on non-convex problems is quite large starting by the results of Bogolyubov [3] and later by Marcellini [12] in one dimension. For $n \geq 2$ we recall Aubert–Tahraoui [2], Mascolo–Schianchi [14–16], Cellina [7], Friesecke [9], Zagatti [19], Sychev [18], Celada–Perrotta [6] and Fonseca–Fusco–Marcellini [8] and Celada–Cupini–Guidorzi [5], through Lipschitz-continuous regularity results for minimizers. We refer to the previous articles for the detailed bibliography on the subject. On the other hand, the uniqueness for non-strictly convex functionals is a classical question and it is however interesting in his own right.

A first uniqueness result is due to Parks [17] which shows the uniqueness of mimimizer for the functional

$$\int_{\Omega} |\nabla u|\, dx \tag{1.3}$$

i.e. $g(t) = t$ provided that the boundary datum satisfies the bounded slope condition. The arguments of Parks's Theorem utilize the fact that since u has the least gradient property, the level sets $E_\lambda = \{x \in \Omega : u(x) \geq \lambda\}$ have the oriented boundary of least area, by the results of Bombieri–De Giorgi–Giusti in [4]. Unfortunately, the elegant approach of Parks does not work for general functionals of type (1.2). Indeed, the integral (1.3) can be reconstructed starting from what happens on the level sets by means of the coarea formula, but for more general non-linear functionals of the form (1.2) the coarea formula does not hold.

A very interesting uniqueness result for non-strictly convex functionals under the assumption

$$g(0) < g(t) \quad \text{for } t > 0 \tag{1.4}$$

is due to Marcellini [13]. In dimension $n \geq 2$, he proved that if G in (1.2) has a minimizer u such that

$$u \in C^1(\overline{\Omega}) \text{ and } Du \neq 0 \text{ everywhere on } \overline{\Omega} \tag{1.5}$$

then u is the unique minimizer of G in the class of all Lipschitz continuous functions that assume the same value of u on $\partial\Omega$. Let us remark that the strict inequality in (1.4) is crucial in order to have uniqueness of the minimizer. Indeed, let us consider the boundary condition constant, say c, then the constant function $u = c$ is a minimizer of G. But, if there exists $t_0 > 0$ such that $g(t) = g(0)$ for any $t \in [0, t_0]$ then for any $\phi \in C_c^\infty(\Omega)$ with $||\nabla\phi||_\infty < t_0$ we get $G(u) = G(u + \phi)$ so that the function $c + \phi$ is still a minimizer of G.

For completeness, we mention also the partial uniqueness result by Kawohl–Stara–Wittum [11].

Inspired by the fundamental contributions of Parks and Marcellini, we show, at least when $n = 2$, that it is possible to remove assumptions (1.5), thus we answer the long standing open question which Marcellini placed in [13]. Precisely, we prove the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and open set and let $g\colon [0, +\infty) \to [0, +\infty]$ be convex and such that (1.4) holds true. Let $G\colon C^{0,1}(\overline{\Omega}) \to \mathbb{R}$ be given by*

$$G(u) := \int_{\Omega} g(|\nabla u(x, y)|)\, dxdy \tag{1.6}$$