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# Uniform convexity, strong convexity and property UC

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Property UC, introduced in 2009, plays a crucial role in several existence and convergence results for best proximity points and fixed points. In this paper, we present two approximation theoretic characterizations of uniform convexity and as consequences of these results, we characterize the uniform convexity in terms of property UC. Characterizations of strong convexity in terms of property UC are also presented.



## 1. Introduction

Let $X$ be a real Banach space. For two non-empty subsets $A$ and $B$ of $X$, we define $d(A,B) = \inf\{\|a-b\| : a \in A \text{ and } b \in B\}$. Taking a cue from [3], the following definition is introduced in [12].

**Definition 1.1.** *A pair $(A,B)$ of non-empty subsets of $X$ is said to satisfy property UC if $\|x_n - y_n\| \to 0$ whenever $(x_n)$ and $(y_n)$ are sequences in $A$ and $(z_n)$ is a sequence in $B$ such that $\|x_n - z_n\| \to d(A,B)$ and $\|y_n - z_n\| \to d(A,B)$.*

The following result is proved in [3].

**Theorem 1.2.** *Consider the following statements.*

(1) *$X$ is uniformly convex.*
(2) *If $A$ and $B$ are non-empty subsets of $X$ such that $A$ is convex then $(A,B)$ has property UC.*

*Then* (1) ⇒ (2).

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The preceding definition and the result are discussed in several papers which deal with existence and convergence results for best proximity points and fixed points (see f.i. [1,3–7,9,10,12]). It is natural to ask whether the condition uniform convexity is necessary in the preceding result. In this paper, we show that the statements (1) and (2) of Theorem 1.2 are in fact equivalent.

Existence of property UC is also studied for some collections of pairs of subsets of spaces which are not uniformly convex [12]. In this paper, we discuss the existence of property UC in strongly convex spaces [8].

The paper is organized as follows. The proof of (2) ⇒ (1) of Theorem 1.2 is presented in Section 3. Section 2 deals with some definitions, notations and elementary results which are needed. In Sections 3, we present two approximation theoretic characterizations of uniform convexity and as consequences of these results, we obtain some characterizations of uniform convexity in terms of property UC. Section 4 deals with two characterizations of strong convexity in terms of property UC.

## 2. Preliminaries

We denote the closed unit ball and unit sphere of $X$ by $B_X$ and $S_X$ respectively. The dual of $X$ is denoted by $X^*$. For a non-empty subset $M$ of $X$, $x \in X$ and $\delta \geq 0$, we define

$$P_M(x, \delta) = \{y \in M : \|x - y\| \leq d(x, M) + \delta\}.$$

Here, $d(x, M) = inf\{\|x - m\| : m \in M\}$. Every element of $P_M(x, 0)$ is called a *best approximation* of $x$ from $M$. If $P_M(x, 0)$ is a singleton for $x \in X$, then $M$ is called *Chebyshev* at $x$. A sequence $(x_n)$ in $M$ is called a *minimizing sequence* for $x \in X$ in $M$, if $\|x - x_n\| \to d(x, M)$. We say that $M$ is *approximatively compact* at $x \in X$, if every minimizing sequence for $x$ in $M$ has a subsequence converging to an element of $M$. If $M$ is approximatively compact and Chebyshev at $x \in X$, then $M$ is called *strongly Chebyshev* at $x$. If $M$ is strongly Chebyshev at every element of a subset $B$, then we say that $M$ is strongly Chebyshev on $B$. We denote the diameter of a bounded subset $A$ of $X$ by $diam(A)$.

The following result follows easily from the definitions.

**Proposition 2.1.** *Let $A$ be a non-empty closed subset of $X$ and $x \in X$. Then the following three statements are equivalent.*

(1) *$A$ is strongly Chebyshev at $x$.*
(2) $diam\Big(P_A(x, \frac{1}{n})\Big) \to 0$.
(3) *$(A, \{x\})$ has property UC.*

We need the following definition and result.

**Definition 2.2.** *Let $M$ and $A$ be non-empty subsets of $X$. The set $M$ is said to be uniformly strongly Chebyshev on $A$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $diam(P_M(x, \delta)) \leq \epsilon$ for every $x \in A$.*

**Proposition 2.3.** *Let $A$ and $B$ be non-empty subsets of $X$. If $A$ is uniformly strongly Chebyshev on $B$, then $(A, B)$ has property UC.*

**Proof.** Let $A$ be uniformly strongly Chebyshev on $B$. Suppose $(A, B)$ does not satisfy property UC. Then there exist $\epsilon_0 > 0$, $(x_n), (y_n)$ in $A$ and $(z_n)$ in $B$ such that $\|x_n - z_n\| \to d(A, B)$, $\|y_n - z_n\| \to d(A, B)$ and $\|x_n - y_n\| > \epsilon_0$ for all $n \in \mathbb{N}$. Since $A$ is uniformly strongly Chebyshev on $B$, there exists $\delta > 0$ such that $diam(P_A(z, \delta)) \leq \epsilon_0$ for all $z \in B$. Observe that there exists $N \in \mathbb{N}$ such that $\|x_n - z_n\| \leq d(A, B) + \delta$ and $\|y_n - z_n\| \leq d(A, B) + \delta$ for all $n \geq N$. Since $d(A, B) \leq d(z_n, A)$ for all $n$, $x_n, y_n \in P_A(z_n, \delta)$ for all $n \geq N$ which is a contradiction. □