J. Math. Anal. Appl. 446 (2017) 1863–1881

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Essential spectrum of the discrete Laplacian on a perturbed periodic graph

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ARTICLE INFO

Article history:
Received 16 October 2015
Available online 5 October 2016
Submitted by P. Exner

Keywords:
Infinite graph
Essential spectrum
Perturbation theory
Discrete Laplacian
Pendant
Random graph

ABSTRACT

We address the Laplacian on a perturbed periodic graph which might not be a periodic graph. We give a criterion for the essential spectrum of the Laplacian on the perturbed graph to include that on the unperturbed graph. This criterion is applicable to a wide class of graphs obtained by a non-compact perturbation such as adding or removing infinitely many vertices and edges. Using this criterion, we demonstrate how to determine the spectra of cone-like graphs, the upper-half plane, and graphs obtained from $\mathbb{Z}^2$ by randomly adding vertices.

1. Introduction

The spectral properties of the Laplacians and Schrödinger operators on various infinite graphs have been studied by numerous authors. For classical results on the spectra of infinite graphs, we refer to a comprehensive survey [16] by Mohar and Woess. Among such studies, covering graphs and periodic graphs are of particular interest due to their applications in mathematics, physics, chemistry, *etc.* In analogy with the case of a Riemannian manifold, Sy and Sunada [22] discussed the bottom of the spectrum of the discrete Schrödinger operator on a covering graph. Motivated by the asymptotic behavior of a random walk, Kotani et al. [14] considered the spectral geometry for an abelian covering graph. Higuchi and Shirai [8] and Higuchi and Nomura [6] studied the spectral properties of the Laplacian on an abelian covering graph. Rabinovich and Roch [19] dealt with the essential spectrum of the Schrödinger operators on $\mathbb{Z}^d$-periodic graphs, motivated by the spectral analysis of quantum graphs. Ando et al. [2] consider the Schrödinger

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http://dx.doi.org/10.1016/j.jmaa.2016.09.063

operators on $\mathbb{Z}^d$-periodic graphs with finite rank perturbations. Korotyaev and Saburova [12] studied the Schrödinger operators with periodic potentials on $\mathbb{Z}^d$-periodic graphs. For related works, we refer to [1,3, 9–11,17,21] and the references therein.

1.1. Main results

In this paper, we consider the essential spectrum of the Laplacian on a perturbed periodic graph, *i.e.*, the perturbed graph is obtained from a $\mathbb{Z}^d$-periodic graph by adding or removing vertices and edges. It is well-known that if the perturbation is compact, the essential spectrum is stable (see Remark 2.3). We are interested in the case in which the perturbation is possibly non-compact, *i.e.*, the operator "$L_{G'} - L_G$" is not a compact operator, where L_G (resp., $L_{G'}$) is the Laplacian on a periodic graph G (resp., a perturbed graph G' of G). If G' is a graph obtained from G by removing and adding some vertices, then G is not a subgraph of G', and *vice versa*. In such cases, the meaning of "$L_{G'} - L_G$" is unclear, because L_G and $L_{G'}$ act on different Hilbert spaces. The precise meaning of "$L_{G'} - L_G$" is given in (2.14). It is noteworthy that a perturbed periodic graph G' might not be periodic. In general, it is difficult to determine the spectrum of an infinite graph, if it does not have a nice symmetry, such as periodicity. In this paper, we present a class of perturbed periodic graphs G' such that the essential spectrum of an unperturbed graph G is contained in that of G':

$$\sigma_{\mathrm{ess}}(L_G) \subset \sigma_{\mathrm{ess}}(L_{G'}). \tag{1.1}$$

We emphasize that the converse of (1.1) cannot be expected in general. As shown in Example 4.4, there exists a perturbed periodic graph G' such that $\sigma_{\mathrm{ess}}(L_G) \subsetneq \sigma_{\mathrm{ess}}(L_{G'})$. In our definition (2.2), L_G is self-adjoint, and its spectrum $\sigma(L_G)$ is contained in $[-1, 1]$. This property raises the question of whether $\sigma(L_G)$ is already all of $[-1, 1]$. In their paper [8], Higuchi and Shirai stated that G has the full spectrum property (FSP) if $\sigma(L_G) = [-1, 1]$, and studied the problem of whether an infinite graph G has the FSP. If (1.1) holds and G has the FSP, then G' has the FSP (see Corollary 3.2). Therefore, it is possible to determine the spectra of perturbed graphs of $\mathbb{Z}^d$, such as those of cones (Example 4.2) and the upper-half plane (Example 4.3). We also discuss the spectrum of a graph obtained from $\mathbb{Z}^2$ by randomly adding pendants, *i.e.*, degree-one vertices connected with an edge.

Relations between the spectrum of the graph Laplacian and those of several operators are known. One of them is the spectrum of the (continuous) Laplacian of a equilateral quantum graph, *i.e.*, the spectrum of the Laplacian on the quantum graph except for a discrete set consists of all real numbers μ for which $\cos\sqrt{\mu}$ is the spectrum of the discrete counterpart of the quantum graph (see [4,13,18] for details). For the (unitary) evolution operator of a quantum walk, called the Grover walk, on a graph, a spectral mapping theorem is known, *i.e.*, the spectrum of such a unitary operator except for ± 1 consists of all complex numbers μ on the unit circle such that $\mathrm{Re}\,\mu$ is the spectrum of the Laplacian on the graph (see [5,7,20]). Our results can be applied to such operators.

1.2. Strategy

Our strategy to establish (1.1) is to construct a Weyl sequence $\{\Psi_n\}$ for the Laplacian $L_{G'}$ such that (i) $\lim_{n\to\infty} \|(L_{G'} - \lambda)\Psi_n\| = 0$ for $\lambda \in \sigma_{\mathrm{ess}}(L_G)$; (ii) $\|\Psi_n\| = 1$; (iii) $\text{w-lim}_{n\to\infty} \Psi_n = 0$. As mentioned above, the Hilbert spaces $\ell^2(V(G))$ and $\ell^2(V(G'))$ on which L_G and $L_{G'}$ act are different (see Section 2 for precise definitions). Generally speaking, there are no inclusion relations between these spaces (see Example 4.2). To give a precise meaning to the perturbation "$L_{G'} - L_G$", we suppose that a vestige of the unperturbed graph G remains in the perturbed graph G'. The vestige is described by a subgraph G'_0 of G', which is isomorphic to a subgraph G_0 of G by an isomorphism φ from G'_0 to G_0. Then, we can naturally define a map $\mathscr{U}_0 : \ell^2(V(G)) \to \ell^2(V(G'))$ as