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A matrix approach to Sheffer polynomials

Lidia Aceto [a,*], Isabel Cação [b,1]

[a] Department of Mathematics, University of Pisa, via F. Buonarroti 1/C, 56127 Pisa, Italy
[b] CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Campus Santiago, 3810-193 Aveiro, Portugal

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ABSTRACT

This paper deals with a unified matrix representation for the Sheffer polynomials. The core of the proposed approach is the so-called creation matrix, a special subdiagonal matrix having as nonzero entries positive integer numbers, whose exponential coincides with the well-known Pascal matrix. In fact, Sheffer polynomials may be expressed in terms of two matrices both connected to it. As we will show, one of them is strictly related to Appell polynomials, while the other is linked to a binomial type sequence. Consequently, different types of Sheffer polynomials correspond to different choices of these two matrices.

1. Introduction

In 1939 I.M. Sheffer in [17] studied in detail the so-called *polynomials sets of type zero* $s_n(x)$, $n \in \mathbb{N}_0$. Given the formal power series f and g such that $f(0) \neq 0$, $g(0) = 0$, $g'(0) \neq 0$, they can be characterized by a generating function of the form $G(x,t) = f(t)\exp(xg(t))$, $x, t \in \mathbb{R}$. Since then, these polynomial sets, nowadays called simply Sheffer polynomials, have been extensively studied not only due to the fact that they arise in various branches of Mathematics but also because of their importance in applied sciences, like Chemistry and Engineering. In the last decades, a renewed interest has been paid to those sequences and to their different representations (see, for example, [10] and references therein). A close connection with Riordan arrays has been established in [11] by proving the isomorphism between the Sheffer group and the Riordan group. Using those results a determinantal approach has been proposed in [19] by extending existent results on determinantal representations of Appell polynomials to Sheffer polynomials (see [8,21]). A generalization of this determinantal approach to mixed special polynomials of two variables related to Gould–Hopper polynomials has been proposed in [13], where properties and operational relations between Sheffer and

* Corresponding author.
E-mail addresses: lidia.aceto@unipi.it (L. Aceto), isabel.cacao@ua.pt (I. Cação).

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those so-called Gould–Hopper–Sheffer polynomials are derived. Moreover, each Sheffer polynomial sequence is related to a binomial type polynomial sequence (cf. [16]) that is proved to be tied to the Bell polynomials, cf. [14,20]. An algebraic approach to the Sheffer polynomial sequences is developed in [9], where the authors also consider matrix representations. In the present paper, we propose an alternative matrix representation to Sheffer polynomials which combines the matrix approach for the representation of Appell polynomials in a real variable, proposed in [2] and recently extended in [3] to the hypercomplex case, with the matrix representation of a binomial type polynomial sequence by the so-called *Bell matrix*. Thus, the matrix which represents the Sheffer polynomial coefficients can be factorized into two matrices, one associated to Appell polynomials and the other linked to the binomial type polynomial sequence, both closely related to the so-called creation matrix introduced in [4]. The result shows clearly the role of Appell and binomial type polynomial sequences into the characterization of Sheffer polynomials. In addition, the special structure of the creation matrix (a subdiagonal matrix which contains as nonzero entries only positive integer numbers) sheds light on the most fundamental arithmetical origins of the class of Sheffer polynomials.

The paper is organized as follows. In Section 2, we present the characterization of Sheffer polynomial sequences by their generating functions and the connection to the Bell polynomials and binomial type polynomial sequences. In Section 3, we propose the matrix approach for the Sheffer polynomials by introducing the generalized Pascal matrix and the Bell matrix and by showing that both are related with the creation matrix. In Section 4, we prove some known properties of Sheffer polynomials by using the proposed matrix representation. In Section 5, we consider some classical Sheffer polynomials as examples and, finally, in Section 6 we present some conclusions.

2. Preliminaries

Let us consider a real numerical sequence $\{b_n\}_{n\geq 0}$, with $b_0 = 0$ and $b_1 \neq 0$ and associate to such numerical sequence the formal power series

$$g(t) = \sum_{n=1}^{+\infty} b_n \frac{t^n}{n!}, \qquad b_1 \neq 0. \tag{1}$$

Then, the function

$$\Phi(x, g(t)) := \exp(xg(t)) = \sum_{n=0}^{+\infty} p_n(x) \frac{t^n}{n!} \tag{2}$$

generates the polynomials $\{p_n(x)\}_{n\geq 0}$ defined as follows [7, Section 3.3, p. 133]

$$p_0(x) = 1, \qquad p_n(x) = \sum_{k=1}^{n} B_{n,k}(b_1, b_2, \ldots, b_{n-k+1}) x^k \quad n \geq 1, \tag{3}$$

where

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$$

are the *(exponential) partial Bell polynomials* in the variables $x_1, x_2, \ldots, x_{n-k+1}$. Explicitly,

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1! \ldots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$